Padé-based numerology

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- Motivation
- Analytic continuation of N_f=2+1+1 QCD lattice data for $T_c(\mu_B^I)$ to $T_c(\mu_B)$
- Multipoint Padé approximants (named Schlessinger point method in Jordi's talk)
- Results with Padé analytic continuation (very preliminary)
- Conclusions

Collaborator/Supervisor: G. Markó **Discussion partners/Advisors:** S. Katz & Sz. Borsányi

Motivation

What can one tell regarding the crossover line between the hadronic and the quark-gluon plasma phases for real μ_q based on lattice data at imaginary μ_q?

An answer is given in R. Bellwied *et al.*, PLB751 (2015) 559-564, arXiv:1507.07510

<u>On the MC simulation:</u>

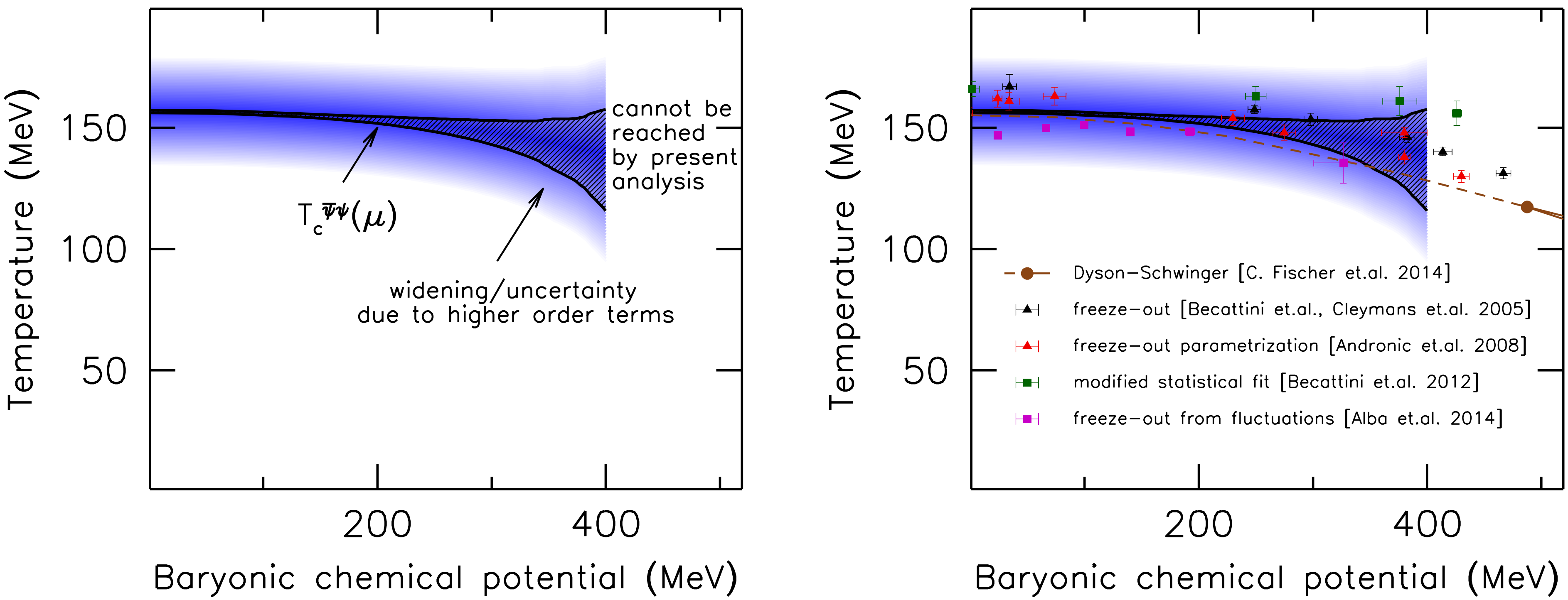
– four times stout smeared staggered action
– dynamical *u, d, s, c* quarks ($m_u = m_d$)
– tree-level Symanzik improvement for gauge action
– LCP: for every lattice spacing a=$0.2\ldots0.063$fm $\frac{m_\pi}{f_\pi}$ and $\frac{m_K}{f_\pi}$ are physical (scale set by f_π)
– simulations performed on $40^3\times10$, $48^3\times12$, $64^3\times16$ lattices at 16 values of T

<u>On the analytical continuation:</u>

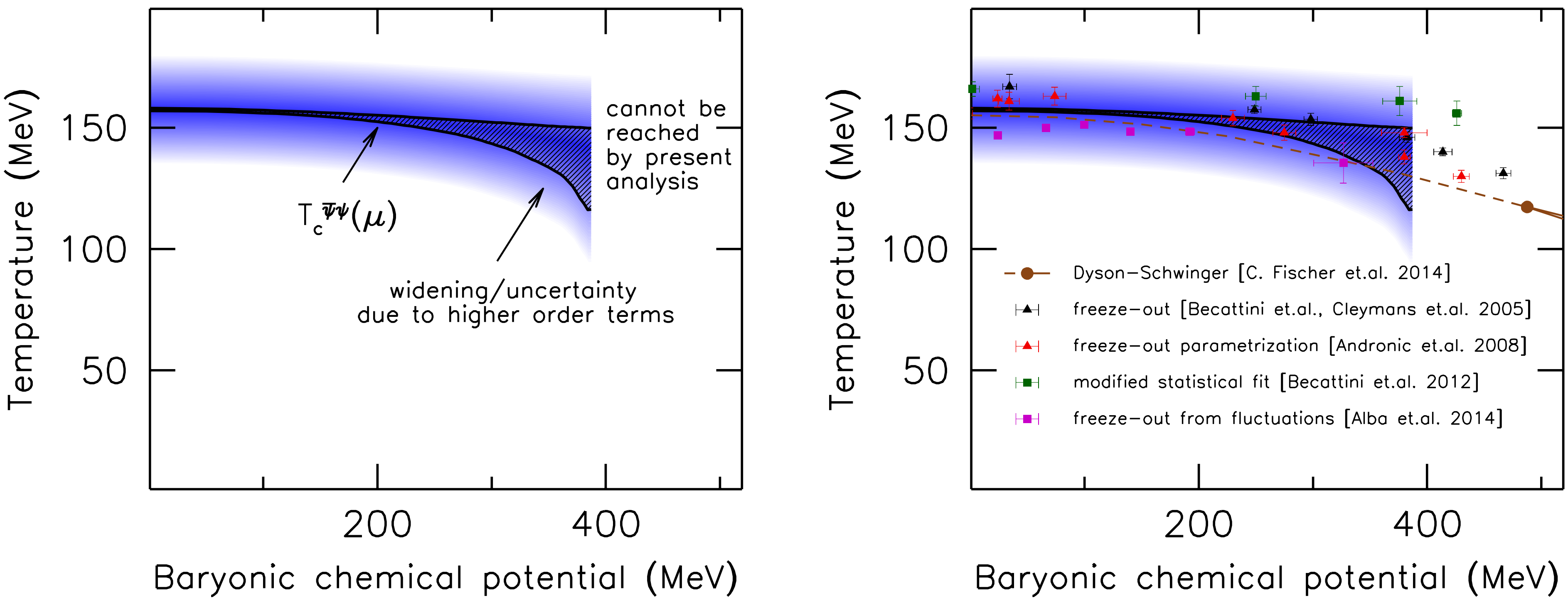
J. Günther's Ph.D thesis (2016) Wuppertal

– data: $T_c^{\bar\psi\psi}$ in MeV for $\hat\mu_{B,I}^{(j)} \equiv \frac{\mu_{B,I}^{(j)}}{T} = j\frac{\pi}{8}$, $\quad j = \{0, 3, 4, 5, 6, 6.5\}$
– fit: $(\hat\mu_{B,I}^{(j)}, T_c)$ using $C_1(x) = 1 + ax + bx^2,\ C_2(x) = \frac{1+ax}{1+bx},\ C_3(x) = \frac{1}{1+ax+bx^2}$
– for real μ_B: solve numerically $\frac{T_c(\mu_B)}{T_c(0)} = C_i\left(\frac{\mu_B^2}{T_c^2(\mu_B)}\right)$ to get $T_c(\mu_B)$
– error analysis using the histogram method

Results presented in R. Bellwied *et al.*, arXiv:1507.07510v1



Results presented in R. Bellwied *et al.*, arXiv:1507.07510v2 (published version)



Numerology refers to a set of believes I have:
– maybe there is more in the lattice data than the authors extracted,
– maybe there is a more educated guess concerning the function to be fitted,
– results from effective models can be used to educate our guess and to indicate which lattice data needs its error reduced
– analytic continuation using Padé approximants can be a tool of choice that can be tested using effecive models

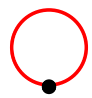
frustration: every other day I think that it is possible to do better, but mostly that it is not possible

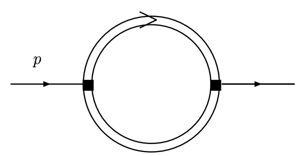
Analytic $T_c(\mu_B)$ and $T_c(\mu_B^I)$ to play with

Obtained in the chiral limit of $N_f = 2$ LσM at LO of $1/N_\pi$ expansion

A. Patkós, Zs. Sz., P. Szépfalusy, A. Jakovác, PLB582 (2004) 179

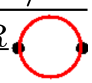
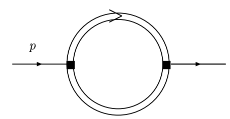
fermions taken into account at lowest order in g (ideal gas)

pion pole: $m^2 + \frac{\lambda}{6}\Phi^2 +$ 

$+$ 

$\Big|_{p^2=M^2} = M^2$

field eq.: $$\left[m^2 + \frac{\lambda}{6}\Phi^2 + \frac{\lambda}{6N}\langle\pi^a\pi^a\rangle\right]\Phi + \frac{g}{N}\langle\bar\psi\psi\rangle = h$$

σ-pole: $G_\sigma^{-1} = p^2 - \frac{h}{\Phi} - \frac{\lambda_R\Phi^2/3}{1-\frac{\lambda_R}{3}\,(\ldots)}$ 
$- \Big[$ 

$- \frac{g}{N\Phi}\langle\bar\psi\psi\rangle\Big] = 0$

localized G_π parameterized with M^2 (self-consistent): $G_\pi(p) = \dfrac{i}{p^2 - M^2}$

analytical continuation: below the $\sigma \to 2\psi$ threshold

$T = \mu = 0 \qquad g = m_q/\Phi = 2m_N/3f_\pi = 6.72$

The phase diagram in the chiral limit, $h = 0$

Gap equation: $M^2[1 - g^2 N_c B(M, m_q)] = 0 \longrightarrow M = 0$ Goldstone theorem

field eq.: $$\Phi\left[m_R^2 + \frac{\lambda_R}{6}\Phi^2 + \frac{\lambda_R}{6} I_{tad}^{\pi}(M=0) - 4N_c \frac{g^2}{\sqrt{N}} I_{tad}^{\Psi}(m_q)\right] = 0$$

analytical determination of the 2nd order line $\Phi(T_c) = 0$: $V_{\text{eff}}(\Phi) = \frac{m_{\text{eff}}^2}{2}\Phi^2 + \frac{\lambda_{\text{eff}}}{4}\Phi^4 + \ldots$

$$0 = m_{\text{eff}}^2 = m_R^2 + \frac{\lambda_R}{72}T_c^2 - \frac{g^2 T_c^2}{2\pi^2} N_c \underbrace{\left(\text{Li}_2(-e^{\mu/T_c}) + \text{Li}_2(-e^{-\mu/T_c})\right)}_{-\frac{\pi^2}{6} - \frac{\mu^2}{2T_c^2}}$$

2nd order line ends when $\lambda_{\text{eff}} = 0$

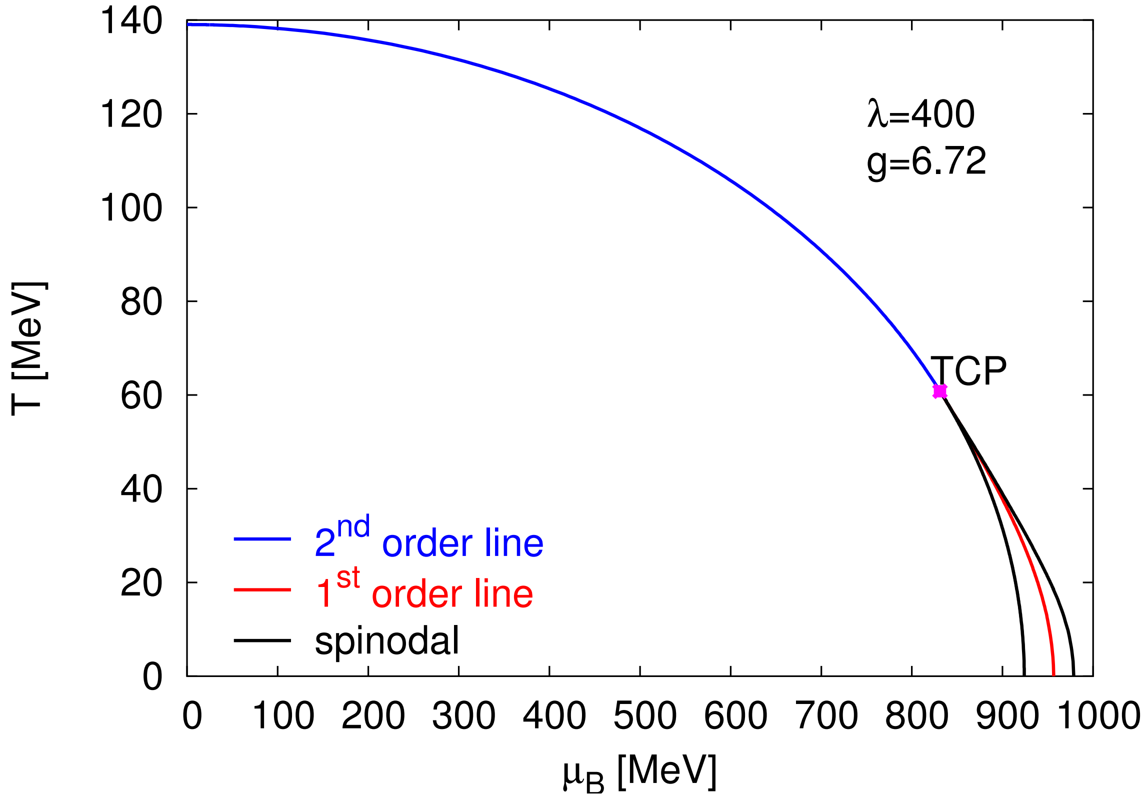
$$\frac{\lambda_R}{6} + \frac{g^4 N_c}{4\pi^2}\left[\frac{\partial}{\partial n}\left(\text{Li}_n(-e^{-\mu/T_c}) + \text{Li}_n(-e^{-\mu/T_c})\right)\Big|_{n=0} - \ln\frac{c T_c}{M_{0B}}\right] = 0$$

$\Rightarrow T_{TCP}$, μ_{TCP}

2nd order line quadratic in μ

$$\text{Li}_{\nu}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^{\nu}}$$

$$\int_0^{\infty} dk \frac{k^{s-1}}{e^{k-\mu} \pm 1} = \pm\Gamma(s)\text{Li}_s(\pm e^{\mu})$$



- $T_c(\mu = 0) = 139$ MeV
- $(T, \mu)_{TCP} = (61, 831)$ MeV

large g $\rightarrow$ strong effect of fermions

- for $h \neq 0$ TCP $\longrightarrow$ CEP softening of the system

  Y. Hatta, T. Ikeda PRD **67** 014028

  even lower value of T_{CEP}

- Z. Fodor, S. D. Katz, hep-lat/0402006

  CEP: $T_E = 162 \pm 2$ MeV, $\mu_E = 360 \pm 40$ MeV $\quad n_f = 2 + 1$

  $T_c(\mu = 0) = 164 \pm 2$ MeV

- $T_c(\mu = 0) = (173 \pm 8)$MeV, 2-flavors Karsch et al, Nucl. Phys. B 605 (2001) 579

- curvature of the 2nd order line: $\frac{T_c}{2} \frac{d^2 T_c}{d\mu^2}\Big|_{\mu=0} = -0.101$

  $-0.07(3)$ lattice result

C.R. Allton et al, Phys. Rev. **D** 68 (2003) 014507

Bottom line: in an ideal gas approximation $T_c(\mu_q)$ is obtained from the equation of an ellipse ($m_R^2 < 0$) in the (T, μ_q) plane

$$m_R^2 + \left(\frac{\lambda_R}{72} + \frac{g^2}{12} N_c\right) T_c^2 + \frac{g^2 N_c}{4\pi^2} \mu_q^2 = 0$$

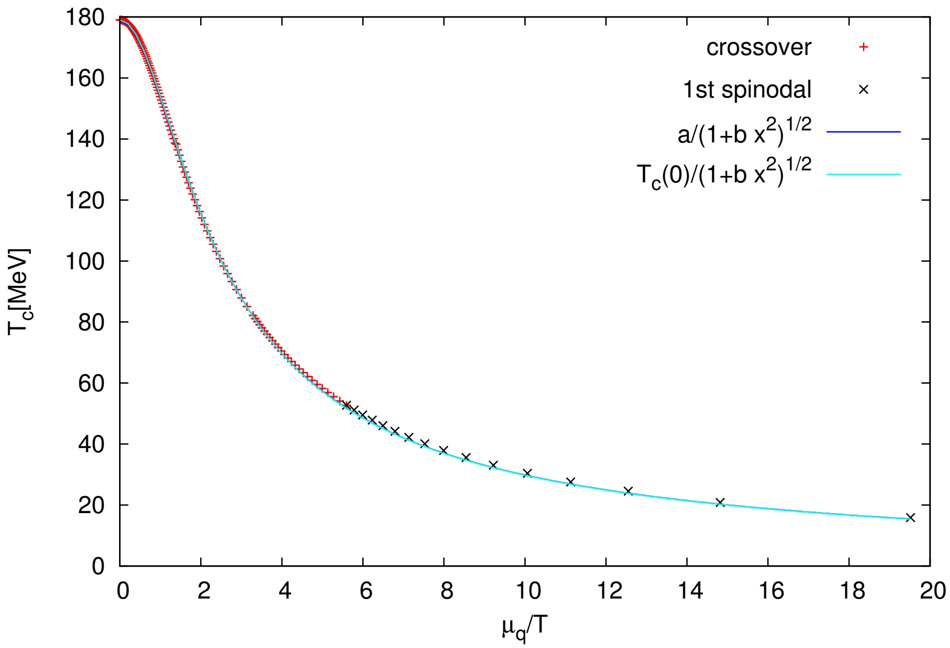
Using the parametrization $\mu_q = j \frac{\pi}{24} T_c$, as in the lattice case

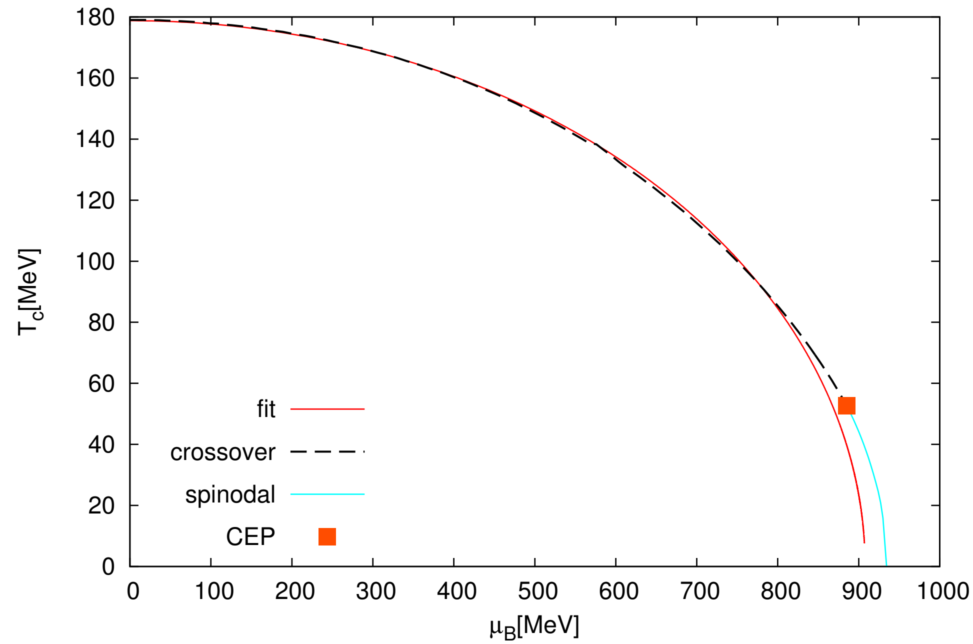
$$T_c(j) = \sqrt{\frac{-m_R^2}{\frac{\lambda_R}{72} + \frac{g^2 N_c}{12}\left(1 \pm \frac{j^2}{192}\right)}}, \quad +/- \text{ : real/imaginary chemical potential}$$

$\Rightarrow$ the most naïve educated guess for analytic continuation is $f(x) = \frac{a}{\sqrt{1+bx}}$, $x = \frac{\mu^2}{T^2}$

Playing with the most naïve educated guess: eLσM, N_f=2+1

Fit on the $(\mu_q, T_c(\mu_q))$ points obtained in P. Kovács, Zs. Sz., Gy. Wolf, PRD93 (2016) 114014

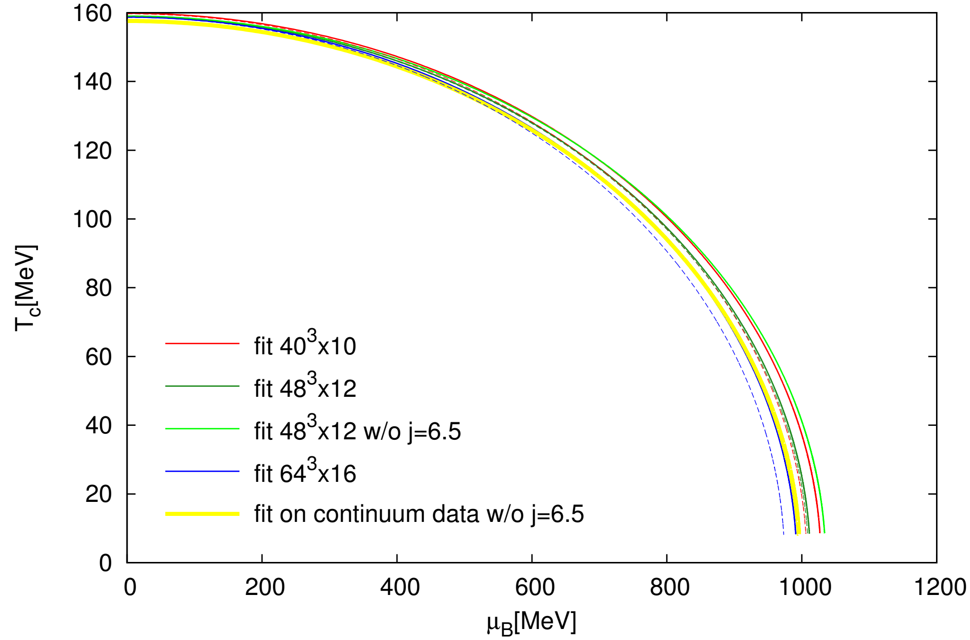




In view of the simple approximation used, there is no surprise that the fit is good.

Could it be that $T_c\left(\frac{\mu_q^2}{T_c^2}\right)$ determined with SDE or FRG shows deviation from $f(x) = \dfrac{a}{\sqrt{1+bx}}$ due to improved resummation and the inclusion of fluctuations ?

Playing with the most naïve educated guess: lattice data



$f(x) = \dfrac{a}{\sqrt{1+bx}}$ fits well to the lattice data, but the unfortunate thing is that the dependence of T_c on $(\mu_q^I/T)^2$ is almost linear $\Longrightarrow$ hard to constrain the functions to be fitted

The error analysis has to be done.

Padé approximant (PA)

Wikipedia: PA is the "best" approximation of a function by a rational function of given order.

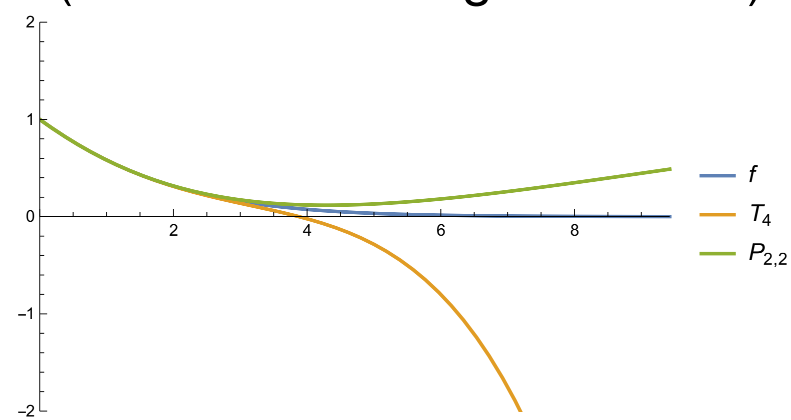
– the (n, m)-th order PA of the function $f(x)$: $P_{n,m}(x) = \dfrac{\sum_{i=0}^{n} a_i x^i}{1 + \sum_{j=1}^{m} b_j x^j}$

– the first n+m Taylor-series coefficients of $f(x)$ and $P_{n,m}(x)$ are the same
$\Longrightarrow$ Taylor expanding around $x = 0$ one has $f(x) - P_{n,m}(x) = \mathcal{O}(x^{m+n+1})$

$$
\begin{aligned}
f(0) &= P_{n,m}(0) \\
f'(0) &= P'_{n,m}(0) \\
f''(0) &= P''_{n,m}(0) \\
&\vdots \\
f^{(n+m)}(0) &= P^{(n+m)}_{n,m}(0)
\end{aligned}
$$

$f(x) = x/(\mathrm{e}^x - 1)$

(radius of convergence $= 2\pi$)



The $n + m + 1$ equations uniquely determine $P_{n,m}(x)$, that is a_i and b_j. Various algorithms exist for the computation of the (n, m)-th order Padé approximant of a function.

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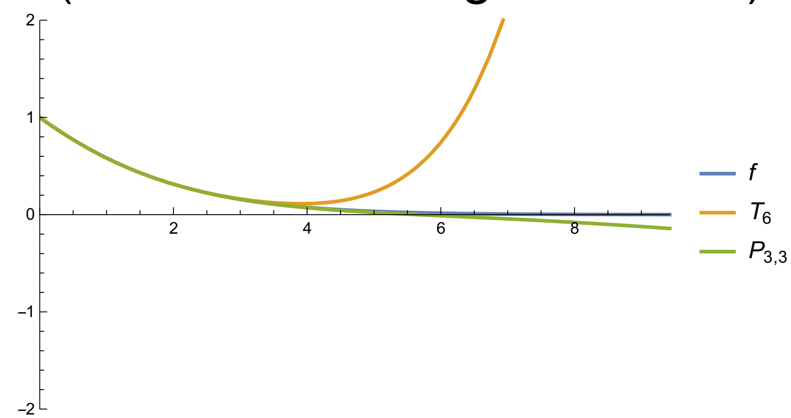

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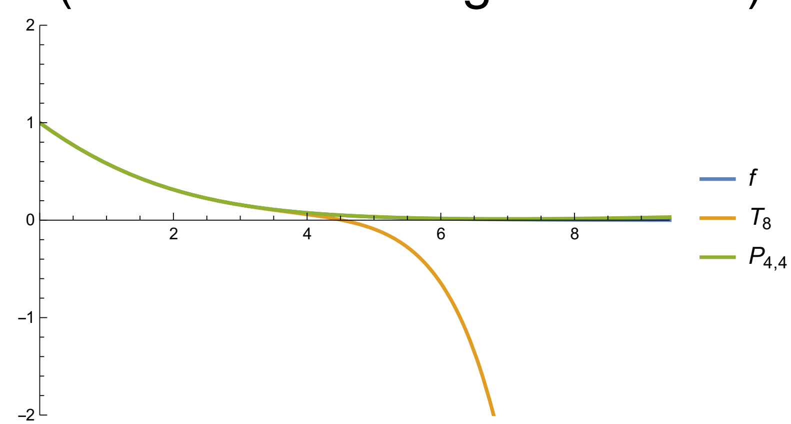

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Multipoint Padé approximants (MPA)

H. J. Vidberg and J. W. Serene, J. Low. Temp. Phys. 29, 179 (1977)

When instead of the N-th derivative of a function at a point one knows the function at $\boldsymbol{N}$ **points** $f_i = f(x_i),\ i = 1\ldots N$ (*e.g.* N Matsubara frequencies), the rational function approximating $f(x)$ is most conveniently given as a **truncated continued fraction**

$$C_N(x) = \frac{a_1}{1+}\,\frac{a_2(x-x_1)}{1+}\cdots\frac{a_N(x-x_{N-1})}{1}\,, \qquad \frac{1}{1+}x \equiv \frac{1}{1+x}$$

- **Task**: find a_i from the conditions $C_N(x_i) = f_i$.
- Can be done **recursively**, by constructing

$$\begin{aligned} g_1(x_i) &= f_i, \quad i = 1, \ldots, N\,, \\ g_p(z) &= \frac{g_{p-1}(x_{p-1}) - g_{p-1}(z)}{(z - x_{p-1})g_{p-1}}, \quad p \geqslant 2\,, \quad z \in \{x_p\} \end{aligned}$$

- $a_i = g_i(x_i)$ are the diagonal elements of an upper triangular matrix, where each row can be determined from the previous one.
- If needed, C_N can be converted into rational form:

  $C_N \longrightarrow P_{\frac{N-1}{2},\frac{N-1}{2}}$ (N odd) or $P_{\frac{N}{2}-1,\frac{N}{2}}$ (N even)

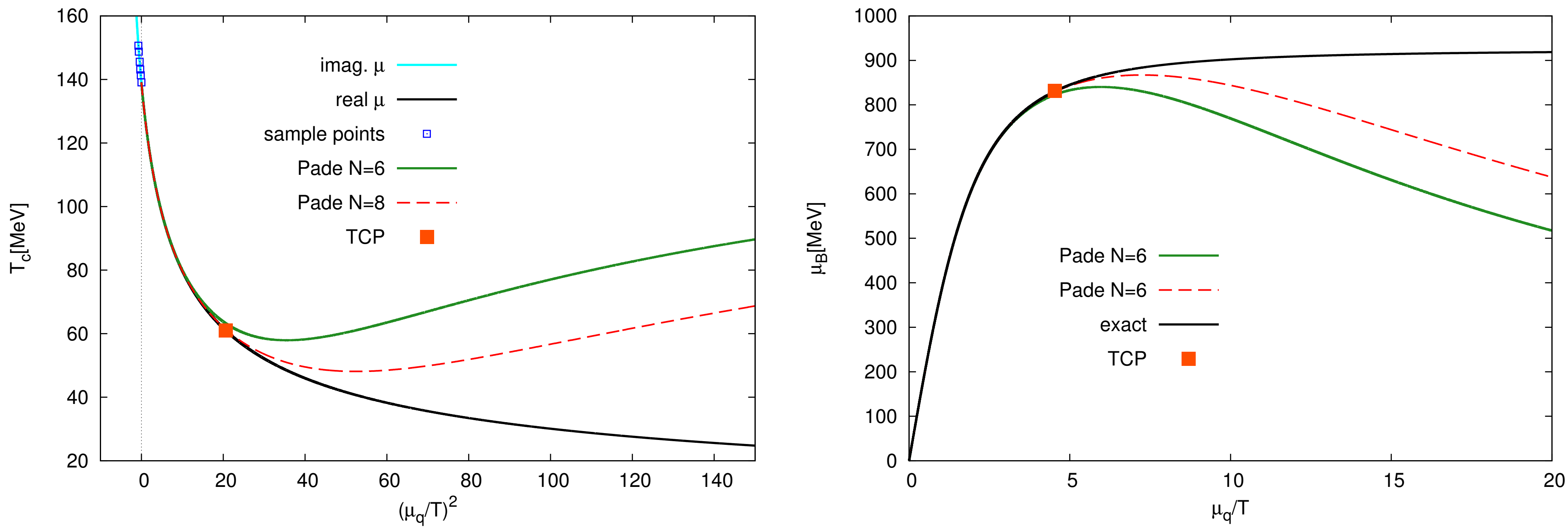
  We work **only** with C_N.

For the continuation of the Eclidean propagator with MPA see also G. Markó *et al.*, PRD96 (2017) 036002

Testing Padé on LσM – 1/3

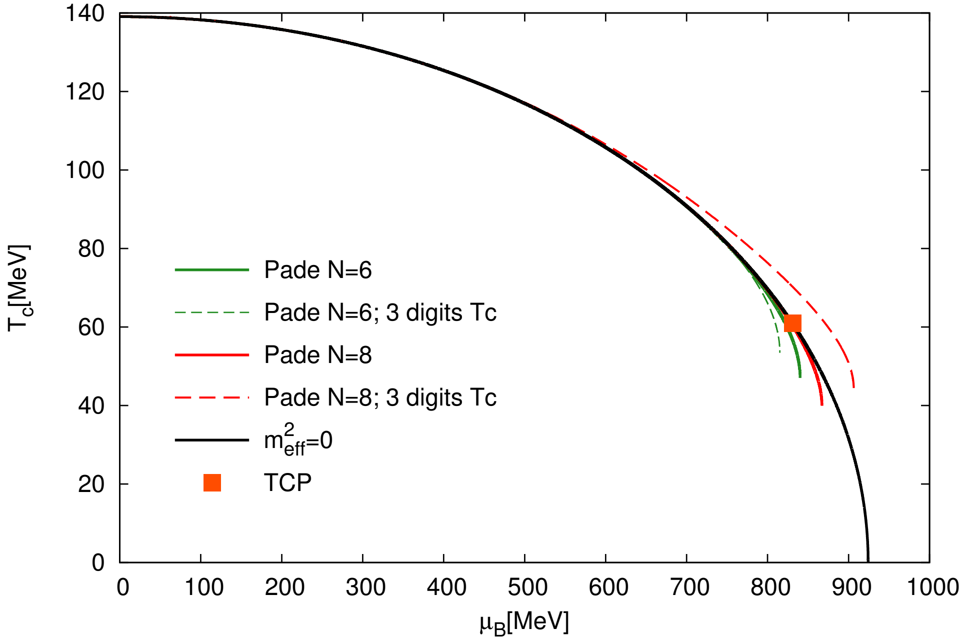
Construct C_N (MPA) using N pairs $(-\hat{\mu}^2_{q,I}, T_c(j))$ and evaluate it for $\hat{\mu}^2_q = \frac{\mu^2_q}{T^2_c}$.

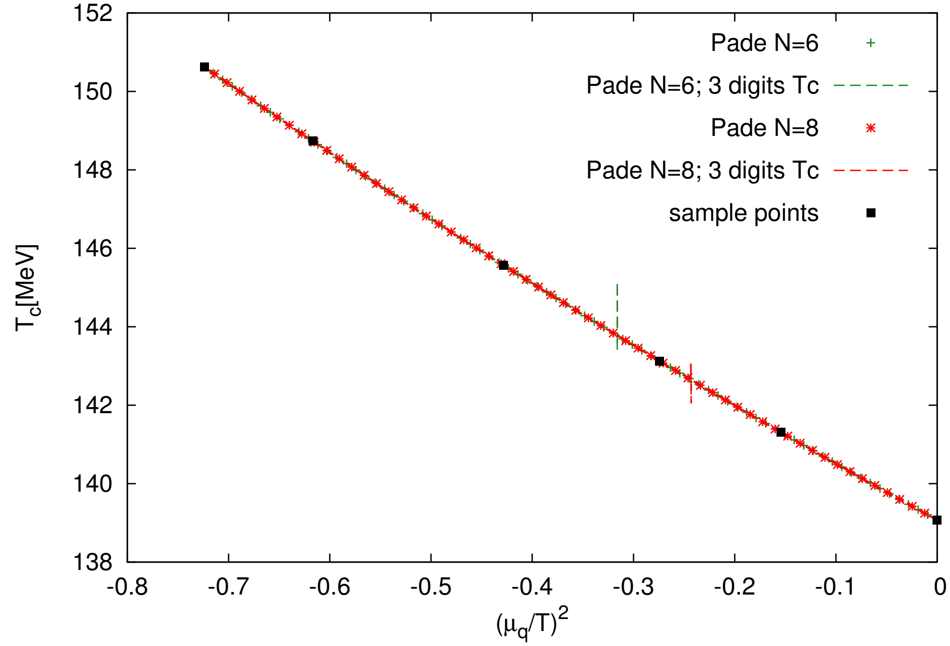
Then $\mu_B = 3\hat{\mu}_q \underbrace{C_N(\hat{\mu}^2_q)}_{T_c^{Pade}}$ and plot the points (μ_B, C_N) to get $T_c(\mu_B)$.



When only points with imaginary μ are included $T_c^{Pade}(\hat{\mu}^2)$ and $\mu_B^{Pade}(\hat{\mu}^2)$ deviate from the exact curve, both have an extremum
$\Longrightarrow T_c^{Pade}(\mu_B)$ is bivalued (the upper part of the curve is relevant).

Testing Padé on LσM – 2/3

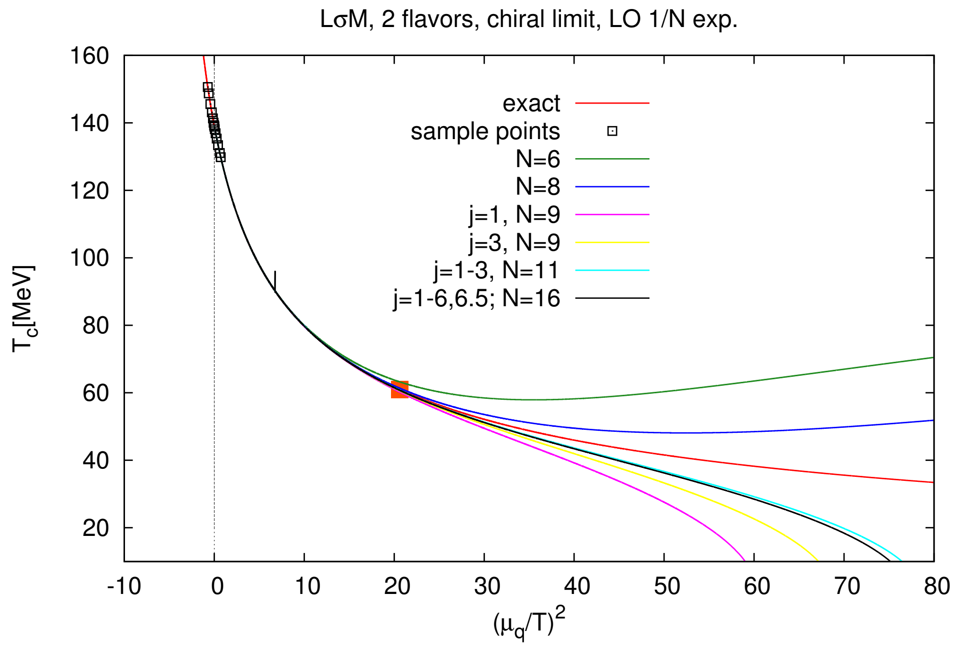


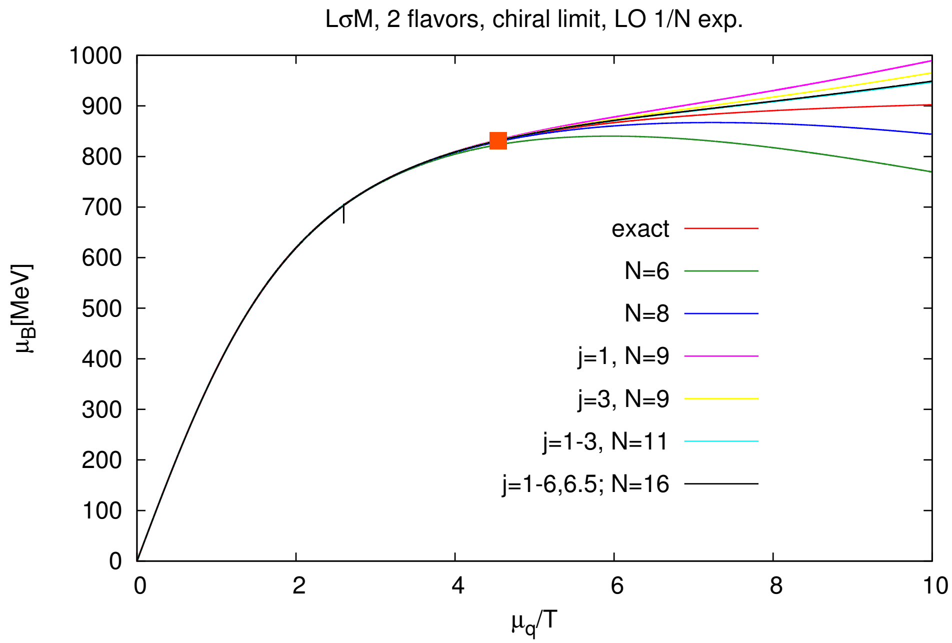


Problem: small variation in the value of T_c can lead to spurious poles which affects the analytic continuation.

Testing Padé on LσM – 3/3

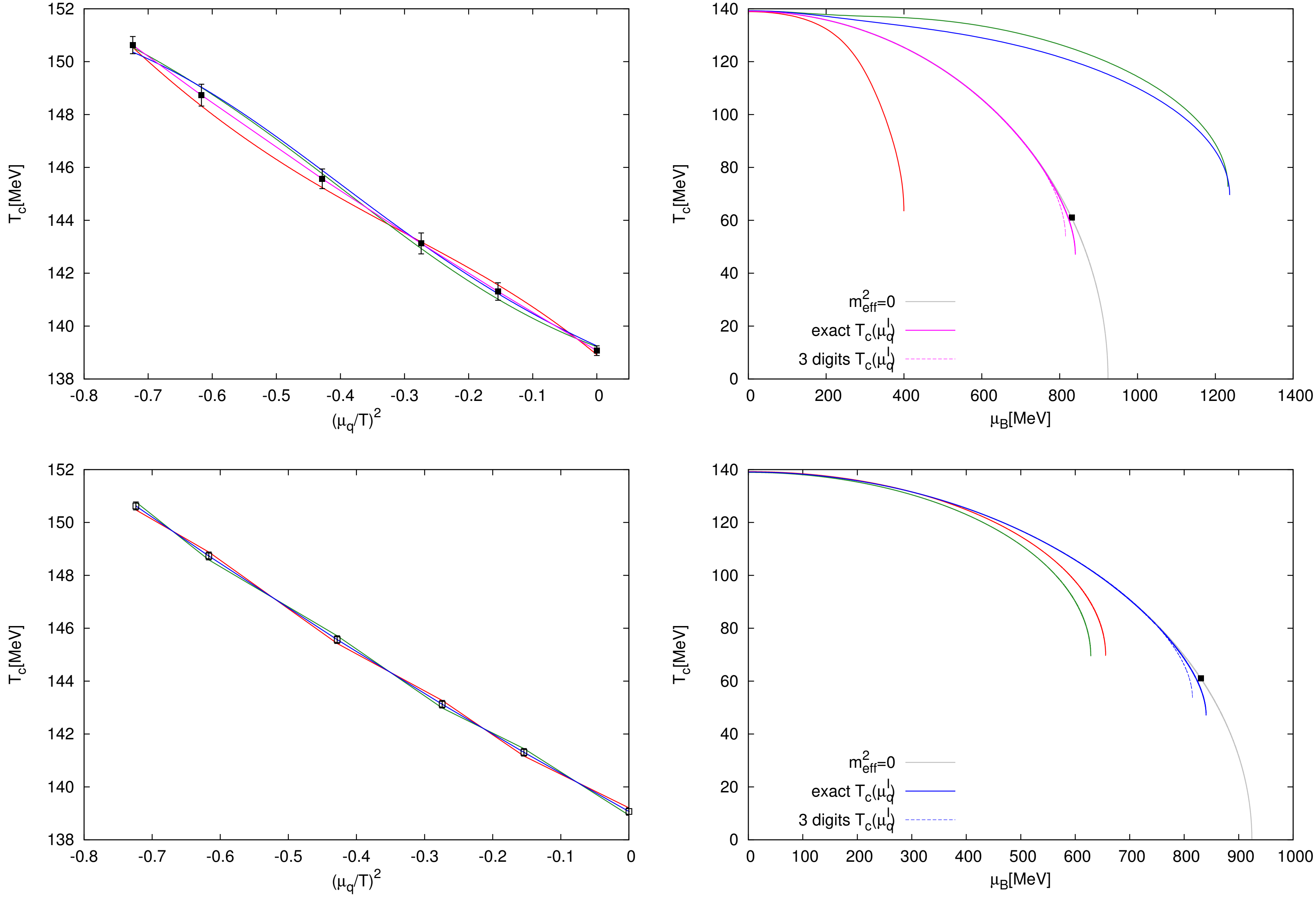
When the multipoint Padé approximant includes points with **real $\boldsymbol{\mu_q}$** an interesting effect is observed: **no extremum** of $T_c^{Pade}(\hat{\mu}^2)$ and $\mu_B^{Pade}(\hat{\mu}^2)$.



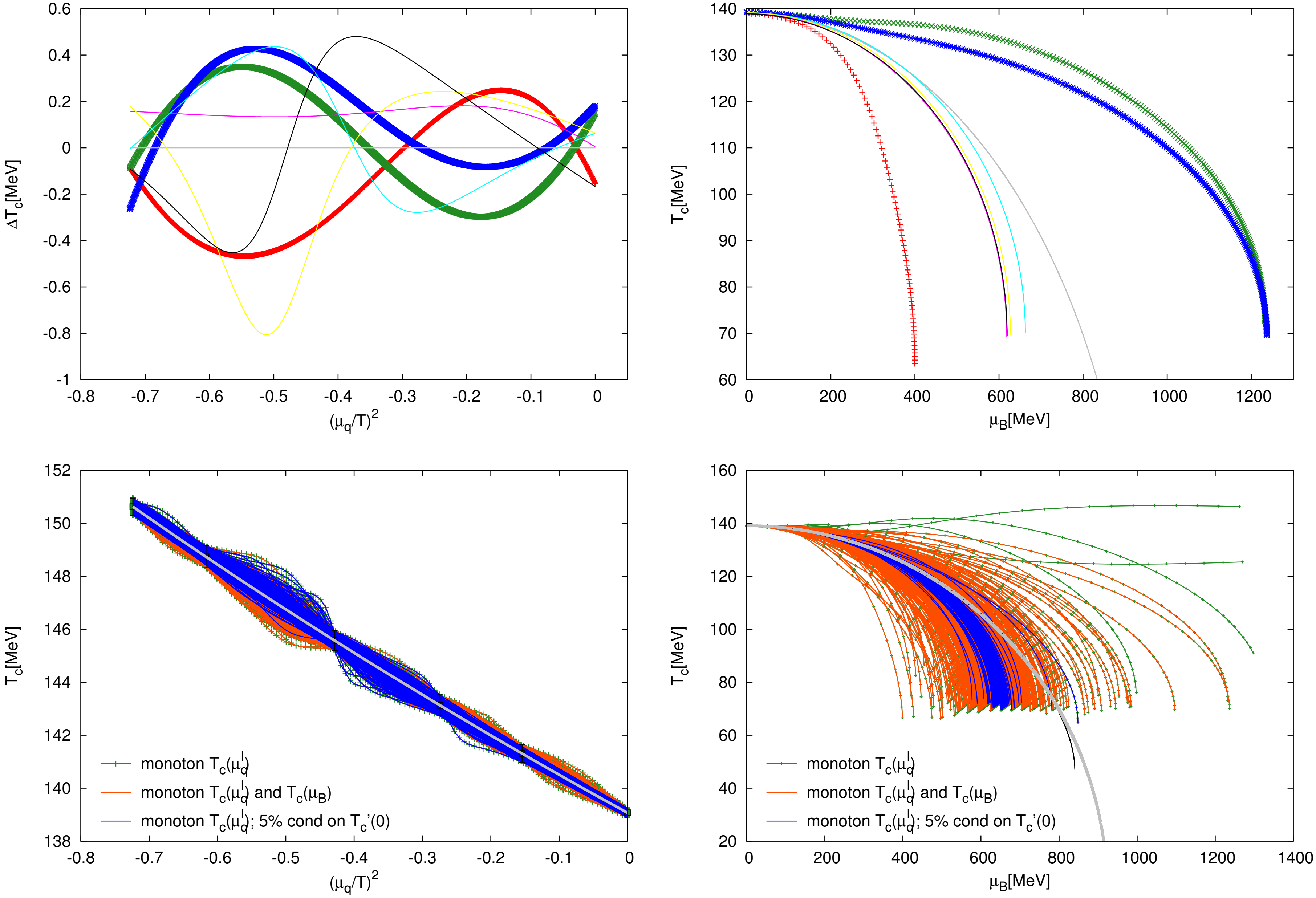


Testing Padé on LσM: including errors – 1/3

Tool: generate values of T_c within the error bounds and use **aggressive intolerance** (the normality/standard is not a priori defined or known, but whatever deviates from it is punished/eliminated a posteriori)

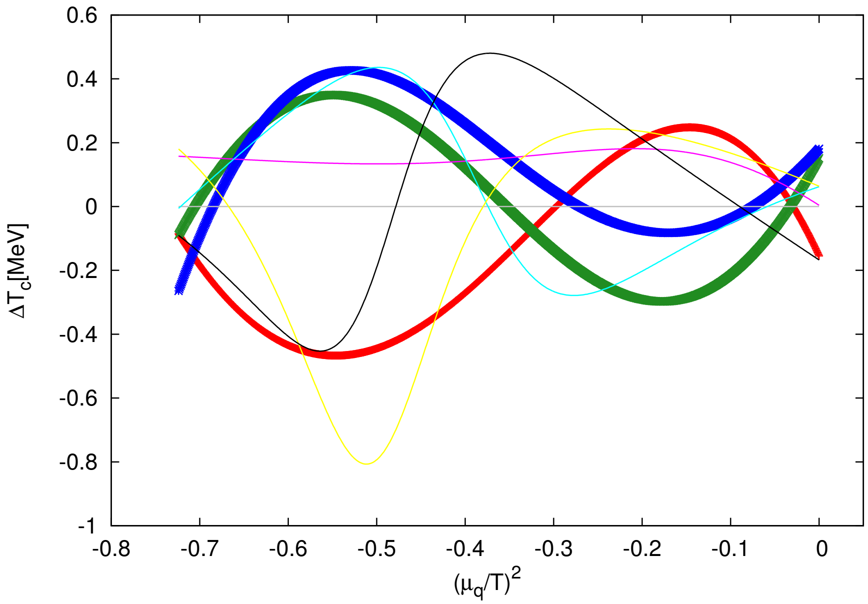


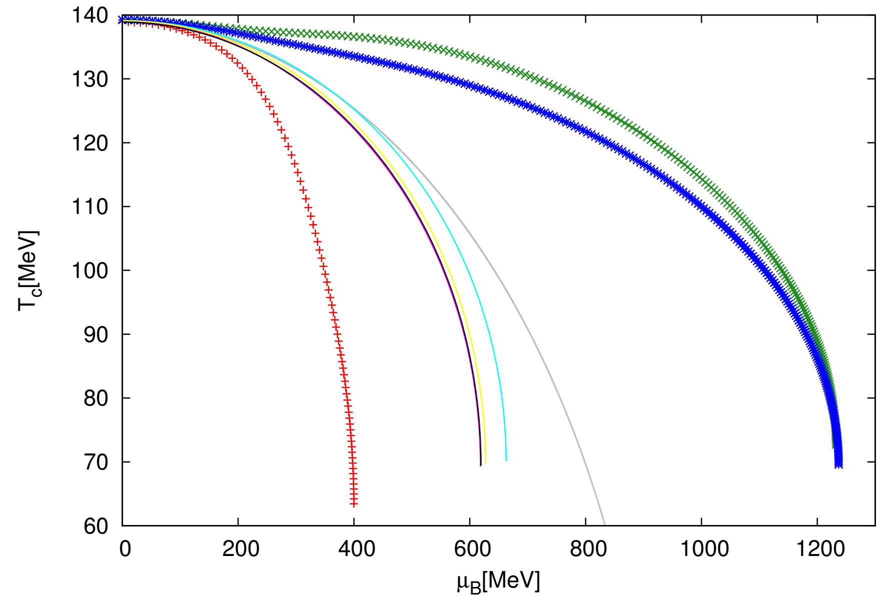
Testing Padé on LσM: including errors – 2/3

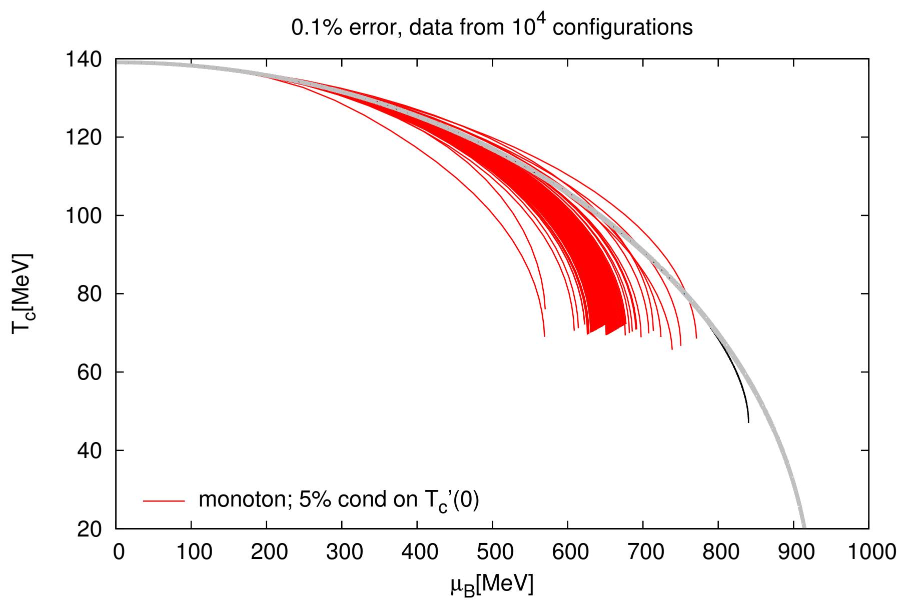


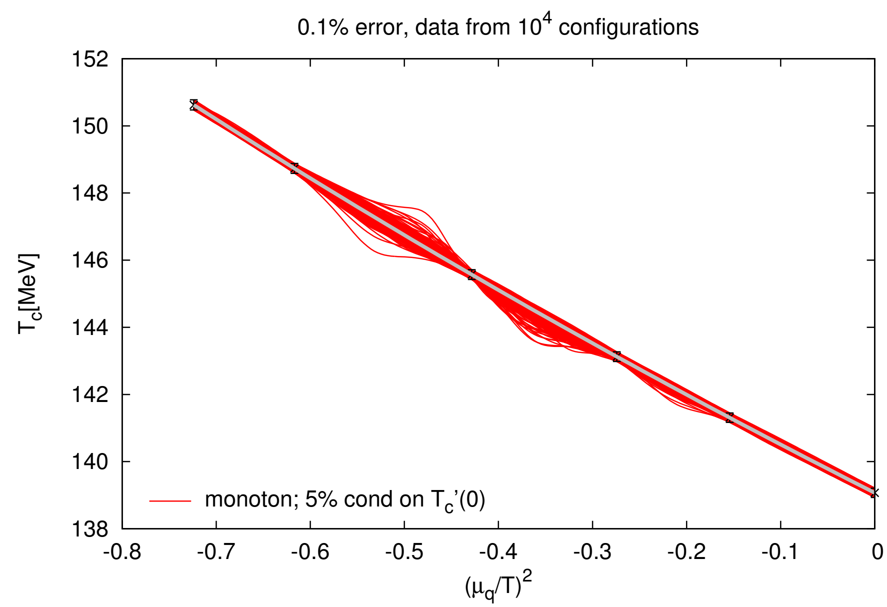
using the error of the lattice data

Testing Padé on LσM: including errors – 3/3



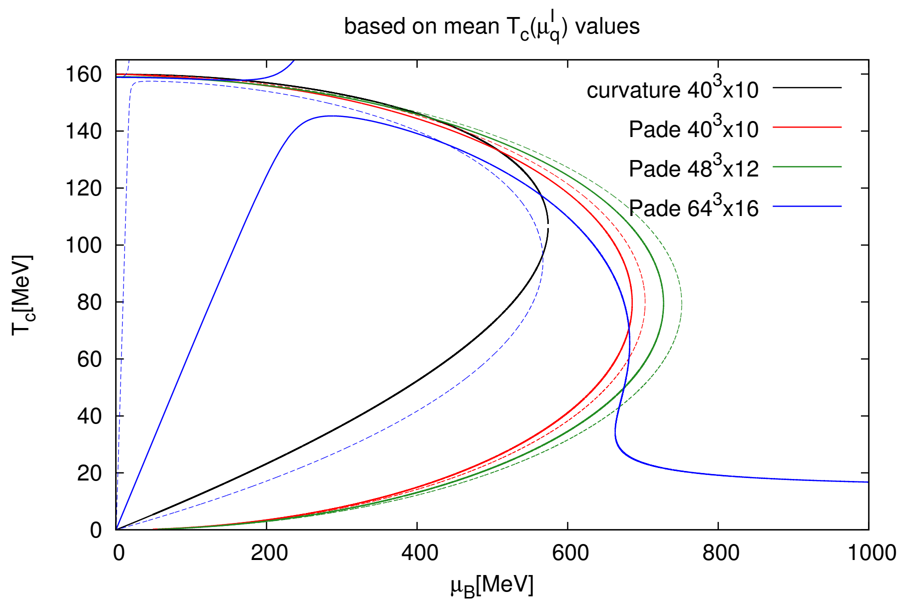


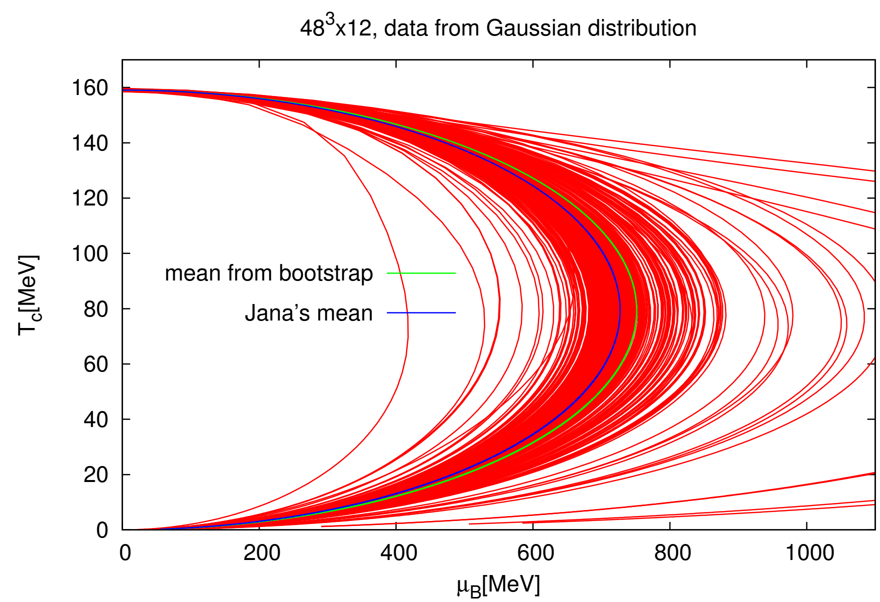




Padé applied to lattice data

Very preliminary results





Summary and Conclusions

- We attempted to locate the crossover line in the $T - \mu_B$ plane with an educated guess for the function used to continue the lattice data at imaginary μ and also with Padé analytic continuation.

- We tested the Padé analytic continuation from imaginary to real μ in an effective model.

- In case of the Padé analytic continuation the difficulty is related to the spurious poles of the Padé approximant.

- We need to develop a method to take into account the error of the lattice data without introducing bias.

- Only after the error is taken into account we can compare with the estimation of $T_c(\mu_B)$ presented in R. Bellwied *et al.,* PLB751 (2015) 559 and asses the usefulness of the Padé approximant to analytically continue lattice data.