Black holes with positive specific heat

$S(E, V)$-thermodynamics

T.S. Biró[1], P.Ván[1], V.G. Czinner[1,2]

[1]Heavy Ion Research Group
MTA Wigner Research Centre for Physics, Budapest

[2]Department for Physics
University of Lisbon, Lisbon

September 18, 2017

What is it About?

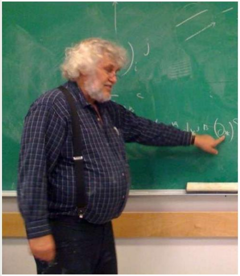
- Hawking radiation, temperature, entropy
- Negative specific heat
- With pressure positive specific heat
- and 4 times the original entropy (at the same temperature)

Outline

Outline

Unruh effect on monochromatic source

Accelerating source → smeared Doppler



Unruh

- **Teljesen klasszikus**
- **Spec. relativitás elég hozzá**

$$I(f) \propto \left| \int e^{i\left[\int \omega \sqrt{\frac{1-v(\tau)}{1+v(\tau)}}\, d\tau - f\tau\right]} d\tau \right|^2$$

Doppler faktor

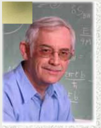
$$I(f) \propto \left| \int_0^\infty e^{ic\omega z/g} z^{-ifc/g-1} dz \right|^2 \propto \frac{1}{e^{2\pi cf/g} - 1}$$

Állandó ‚g' gyorsulás együttmozgó rendszerben: $dv/d\tau = -g(1-v^2)$

Bekenstein-Hawking entropy

Simple estimate

Entropy = 1/4 Horizon Area



At Schwarzschild radius $R = 2GM/c^2$ the gravitational acceleration is $g = GM/R^2 = c^2/2R$. The Unruh temperature in proper units becomes

$$k_B T_U = \frac{\hbar g}{2\pi c}. \tag{1}$$

Clausius' entropy if $E = Mc^2$ becomes

$$\frac{S}{k_B} = \int \frac{d(Mc^2)}{k_B T} = \frac{c^3}{\hbar G}\pi R^2 = \frac{1}{4}\,\frac{A}{L_P^2}. \tag{2}$$

Temperature

Unruh, Hawking, spectral

Planck scale units: $c = 1$, $L_P = GM_P$, $L_P = \hbar/M_P$. Entropy units: $k_B = 1$.
(constant) acceleration occurs as (constant) temperature

$$T_U(g) = \frac{g}{2\pi}. \tag{3}$$

Radial space time metric

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)}, \tag{4}$$

Maupertuis action $S = m \int ds = \int L ds$ in terms of proper time. Being a static metric, $E = \frac{\partial L}{\partial \dot{t}} = f(r)\dot{t}$ is constant. Therefore

$$\dot{r}^2 = K^2 - f(r). \tag{5}$$

Its total derivative delivers the comoving aceleration

$$\ddot{r} = -\frac{1}{2} f'(r) \tag{6}$$

Entropy to Unruh and Hawking temperatures

Size of horizon

Horizon at $r = R$ with $f(R) = 0$. Here $g = |f'(R)|/2$.

$$T_H = T_U\left(\frac{1}{2}\left|f'(R)\right|\right) = \frac{1}{4\pi}\left|f'(R)\right|. \tag{7}$$

For the Schwarzschild metric $f(r) = 1 - 2M/r$, $R = 2M$ and $f'(R) = 1/2M$

$$T_H = \frac{1}{8\pi M}. \tag{8}$$

Clausius' entropy:

$$S = \int \frac{dE}{T} = 4\pi \int \frac{dM}{|f'(R)|} = 4\pi \int\int \delta(f(r,M))\, drdM. \tag{9}$$

As a volume integral it defines entropy density:

$$S = \int s\, d^3r = \int \left[\int\int \delta\left(r^2 f(r,M)\right)\, dM\right] d^3r. \tag{10}$$

for Schwarzschild $S = \pi R^2 = A/4$.

Horizon entropy thermodynamics

negative specific heat

Traditional analysis considers $E = M$ and the EoS

$$S = \pi R^2 = 4\pi E^2. \tag{11}$$

Derivatives ☺

$$\frac{1}{T} = \frac{\partial S}{\partial E} = 8\pi M = \frac{1}{T_H}. \tag{12}$$

Instability problem: ☹

$$-\frac{1}{V c_V T^2} = \frac{\partial^2 S}{\partial E^2} = 8\pi > 0 \tag{13}$$

leads to $c_V < 0$ and instable entropy maximum.

Outline

$S(E, V)$ thermodynamics

$E = M$, $R = 2M$ put at the end only

Scaling anstaz: $S(E, V) \; = \; V\,\mathfrak{s}(E/V)$.
Energy density: $\varepsilon = E/V$ in the above ansatz.

Homogeneous EoS, $\mathfrak{s}(\varepsilon)$, based thermodynamics.

$$\frac{1}{T} \quad = \quad \frac{\partial S}{\partial E} = \mathfrak{s}'(\varepsilon),$$
$$\frac{p}{T} \quad = \quad \frac{\partial S}{\partial V} = \mathfrak{s}(\varepsilon) - \varepsilon\mathfrak{s}'(\varepsilon). \tag{14}$$

This satisfies $T\mathfrak{s} = \varepsilon + p$

EoS from radial integrals

$$S = \int_0^{2M} \mathfrak{s}(r)\, 4\pi r^2\, dr = aS_H = 4\pi a M^2,$$

$$E = \int_0^{2M} \varepsilon(r)\, 4\pi r^2\, dr = M, \tag{15}$$

results in

$$\mathfrak{s} = a/2r, \qquad \varepsilon(r) = 1/8\pi r^2, \tag{16}$$

to conclude at the equation of state

Scaling Horizon Equation of State

$$\mathfrak{s}(\varepsilon) = a\sqrt{2\pi\varepsilon}. \tag{17}$$

Consequences of scaling EoS

Hawking radiation pressure

From the $\mathfrak{s}(\varepsilon) = \kappa\sqrt{\varepsilon}$ type EoS it follows

$$\frac{1}{T} = \frac{\mathfrak{s}}{2\varepsilon} = \frac{S}{2E}, \qquad \text{and} \qquad \frac{p}{T} = \frac{\mathfrak{s}}{2}. \tag{18}$$

Scaling Horizon Equation of State

$$p(\varepsilon) = \varepsilon. \tag{19}$$

1-dim ideal radiation, causal

Outline

Consequences of scaling EoS

Thermal stability

$$\det \begin{pmatrix} \frac{\partial^2 S}{\partial E^2} & \frac{\partial^2 S}{\partial E \partial V} \\ \frac{\partial^2 S}{\partial V \partial E} & \frac{\partial^2 S}{\partial V^2} \end{pmatrix} = \frac{\mathfrak{s}''(\varepsilon)}{V} \det \begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon^2 \end{pmatrix} = 0. \tag{20}$$

Heat capacity and specific heat

$$\frac{\partial^2 S}{\partial E^2} = \frac{\partial}{\partial E} \mathfrak{s}'(\varepsilon) = \frac{1}{V} \mathfrak{s}''(\varepsilon). \tag{21}$$

$$\frac{\partial^2 S}{\partial E^2} = \frac{\partial}{\partial E} \frac{1}{T} = -\frac{1}{T^2} \frac{\partial T}{\partial E} = -\frac{1}{C_V T^2}. \tag{22}$$

Stable if $\frac{\partial^2 S}{\partial E^2} < 0$, i.e. with **positive** C_V.

Positivity of specific heat

$c_V = C_V/V$ (heat capacity per volume)

$$c_V = -\frac{1}{T^2 \mathfrak{s}''(\varepsilon)} = -\frac{\mathfrak{s}'(\varepsilon)^2}{\mathfrak{s}''(\varepsilon)} \tag{23}$$

In our case with horizons $\mathfrak{s}'' = -\mathfrak{s}/4\varepsilon^2$. It delivers

$$c_V = -\frac{(\mathfrak{s}/2\epsilon)^2}{(-\mathfrak{s}/4\varepsilon^2)} = \mathfrak{s} = a\sqrt{2\pi\varepsilon} > 0. \tag{24}$$

Comparison with Hawking

Hawking	Present
$S_H = 4\pi M^2$	$S = 4\pi a M^2$
$1/T_H = 8\pi M$	$1/T = 2\pi a M$
$p_H = 0$	$p = \varepsilon = 3/32\pi M^2 = 3/2A$
$c_V^{(H)} = -3/4M < 0$	$c_V = S/V = 3a/8M > 0$

Which choice to make?

$a = 4$

Relation to Hawking temperature (another view):

$$\frac{1}{T_H} = \frac{dS_H}{dM} = \frac{1}{a}\left[1 \cdot \frac{\partial S}{\partial E} + \frac{dV}{dM} \cdot \frac{\partial S}{\partial V}\right]$$

$$= \frac{1}{a}\left[\frac{1}{T} + 2\,\frac{p}{T}\,\frac{dV}{dR}\right] = \frac{1+2pA}{aT} = \frac{4}{aT}. \tag{25}$$

By the particular choice $a = 4$ we have

$$p = \varepsilon = \frac{3}{2A}, \qquad T = T_H, \qquad S = 4S_H = A, \qquad c_V = 3/2M = -2c_V^{(H)}. \tag{26}$$

Proposal: *black hole entropy is the total horizon area* (in Planck units).

Outline

Horizon around static charge Q and mass M

Reissner-Nordstrøm metric

$r^2 f(r) = r^2 - 2Mr + Q = (r - r_+)(r - r_)$ with

$$r_\pm \;=\; M\left(1 \pm \sqrt{1-\mu^2}\right) \tag{27}$$

with $\mu = Q/E = Q/M$.

We seek for an entropy formula

$$S(E,V,Q) \;=\; \kappa(\mu)\sqrt{EV} \;=\; a\pi r_+^2. \tag{28}$$

We require $E = M$ and $\Phi = Q/r_+$ Coulomb potential.

Thermodynamics of charged black holes

The $S(E, V, Q)$ view

$$\begin{aligned}
\frac{1}{T} &= \left.\frac{\partial S}{\partial E}\right|_{V,Q} &= \frac{1}{2}\sqrt{\frac{V}{E}}\,\kappa\left(1-\frac{2\mu\kappa'}{\kappa}\right), \\
-\frac{\Phi}{T} &= \left.\frac{\partial S}{\partial Q}\right|_{E,V} &= \frac{1}{2}\sqrt{\frac{V}{E}}\,\kappa\left(\frac{2\kappa'}{\kappa}\right), \\
\frac{p}{T} &= \left.\frac{\partial S}{\partial V}\right|_{E,Q} &= \frac{1}{2}\sqrt{\frac{E}{V}}\,\kappa
\end{aligned} \tag{29}$$

This satisfies $TS = E + pV - \Phi Q$

Charged black holes

Determination of $\kappa(\mu)$

We identify the Coulomb potential: $\Phi = Q/r_+$:

$$\Phi = -\frac{-\Phi/T}{1/T} = \frac{\frac{2\kappa'}{\kappa}}{\frac{2\kappa'}{\kappa}\mu - 1}, \tag{30}$$

$$\frac{Q}{r_+} = \frac{\mu}{1+\sqrt{1-\mu^2}}. \tag{31}$$

The solution is

$$\kappa(\mu)^2 = \mathrm{const} \times \left(1+\sqrt{1-\mu^2}\right) \propto r_+/M. \tag{32}$$

The constant from $V = 4\pi r_+^3/3$ is $3\pi a^2/4$.

Chraged black hole

pressure at the horizon

$$p = \frac{p/T}{1/T} = \frac{\varepsilon}{1 - \mu^{\frac{2\kappa'}{\kappa}}} \tag{33}$$

Using the above solution for $\kappa(\mu)$ we arrive at the EoS:

$$p = \varepsilon\sqrt{1 - \mu^2}. \tag{34}$$

For $\mu = 0$ we are back to the Schwarzschild result: $p = \varepsilon$.

For extremely charged black holes $\mu = 1$, the pressure vanishes: $p = 0$.

They have also $T_H = 0$, so this is good...

Comparison with Hawking

The temperatures

"Old" construction (counting with temperature):

$$\frac{1}{T_H} = \frac{\partial}{\partial M} S_H(M,Q) = \frac{2S}{\sqrt{M^2-Q^2}} \tag{35}$$

"New" construction (counting also with pressure) by taking $E = M$, $r_+(M)$ and $S = a\pi r_+^2$:

$$\frac{1}{T} = \frac{\partial}{\partial E} S(E,V,Q)\bigg|_{V,Q} = \frac{S}{2\sqrt{M^2-Q^2}} = \frac{a}{4}\,\frac{1}{T_H} \tag{36}$$

Heat capacity

$$-\frac{1}{CT^2} = \frac{\partial}{\partial E}\frac{1}{T}\bigg|_{V,Q} = \frac{S}{4(M^2-Q^2)^{3/2}}\left(\sqrt{M^2-Q^2}-2M\right) < 0. \tag{37}$$

Thermal stability

Specific heat vs geometry

We express our results in terms of $r_\pm$:

$$C = a\pi r_+^2 \frac{r_+ - r_-}{r_+ + 3r_-} > 0. \tag{38}$$

$$T = \frac{r_+ - r_-}{a\pi r_+^2}. \tag{39}$$

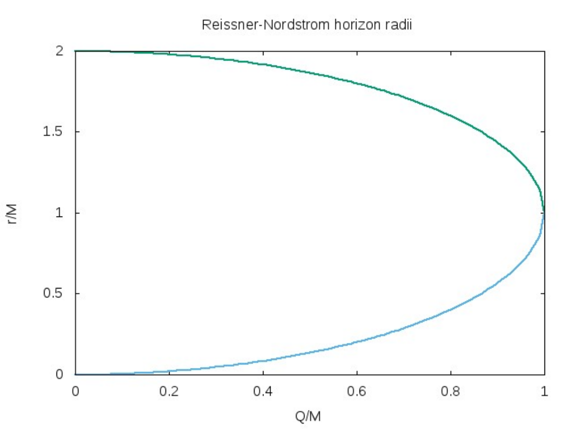
Specific heat behaves well at $T = 0$ (3rd law):

$$c_{V,Q} = \frac{C}{V} = \frac{3\pi a^2}{4} \frac{r_+}{r_3 + 3r_-} T. \tag{40}$$

Note: still $S \neq 0$ at $T = 0$.

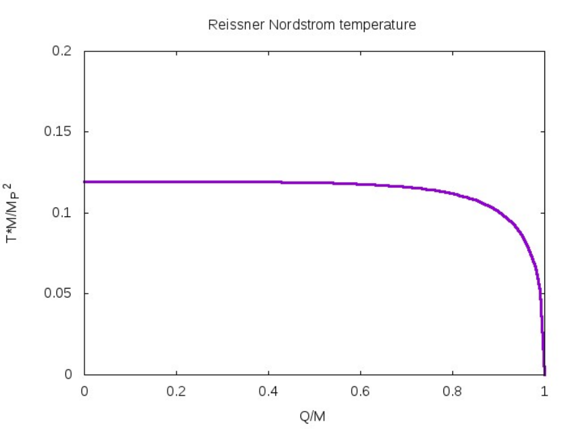
Charged black hole thermodynamics

in pictures 1



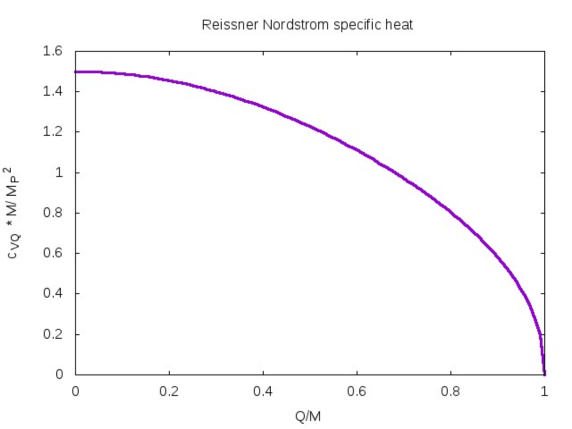
Charged black hole thermodynamics

in pictures 2



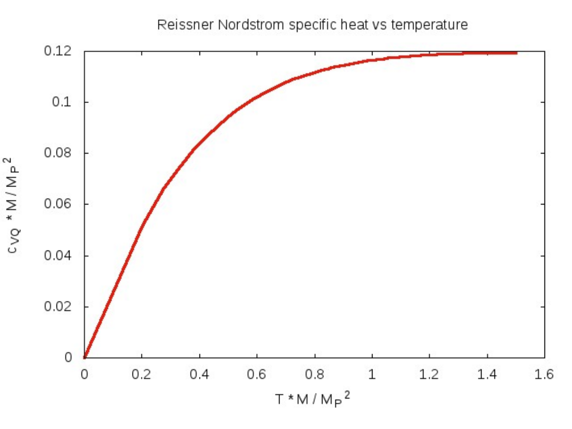
Charged black hole thermodynamics

in pictures 3



Charged black hole thermodynamics

in pictures 4



Summary

- $S(E, V)$ thermodynamics instead of $S(E)$ thermodynamics
- Hawking radiation has not only temperature, but also pressure
- The horizon area shall be a better entropy measure than its one fourth