Properties of bound states from the FRG and the DSE-BSE approaches

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- Brief introduction to the FRG
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- Results in the Quark-Meson model
- Preliminary comparison with the DSE-BSE approach
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Motivation

Observable properties of hadrons are difficult to extract from QCD's degrees of freedom:

- Need theoretical assumptions:
- Bound states and QCD (at hadronic energies) are not perturbative.

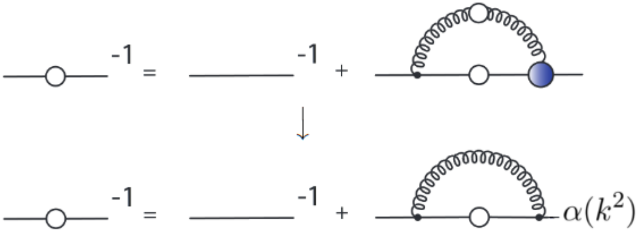
Many approaches and models are built to solve these problems.

- Lattice QCD.
- Functional methods.
  - **Dyson-Schwinger equations (DSE)**.
  - **Bethe-Salpeter equations (BSE)**.

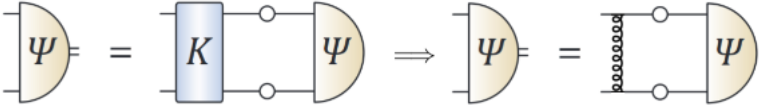
Both equations need to be truncated to be numerically solved.

- Rainbow-Ladder truncation:

Quark propagator:



Bethe-Salpeter equation:



Results for the pion and other ground state mesons are well understood. However:

- The solution relies strongly on the truncation.
- For more complex systems:
    - Other terms appear in the DSE.
    - Rainbow-Ladder truncation is not good enough.
    - New technical issues appear.

Work in a different approach:

- Use of the **Functional Renormalization Group (FRG)** to find properties of mesons.
- The FRG approach is consistent with BSE.

Introduction to the FRG

Generating Functional in Euclidean space as starting point:

$$\mathcal{Z}[J] = e^{W[J]} = \int \mathcal{D}\psi \; e^{-S[\psi]+\int_x J\psi}$$

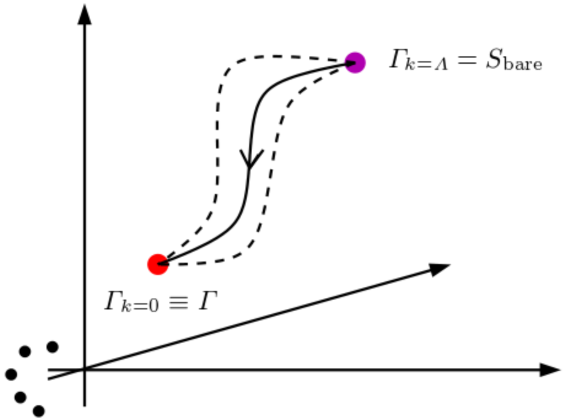
Effective Action $\Gamma[\phi]$ from $W[J]$ Legendre transformation:

$$e^{-\Gamma[\phi]} = \int \mathcal{D}\psi \; \exp\left(-S[\phi+\psi] + \int_x \frac{d\Gamma[\phi]}{d\phi}\psi\right)$$

with $\frac{\delta\Gamma}{\delta\psi} \equiv J$, $\phi \equiv \frac{\delta W[J]}{\delta J} = \langle\psi\rangle_J$. $\Gamma[\phi]$ expressed as sum of 1PI diagrams.
Introduction of scale k and regulator ΔS_k:

$$\Gamma_k[\phi] = \Gamma[\phi] - \Delta S_k[\phi]$$

- Initial and final conditions are fixed in theory space:



- The choice of the regulator is not unique.

With the choice of quadratic regulators $\Delta S_k[\phi] = \int_p \phi R_k \phi$, the properties of the scale-dependent effective action lead to the **1-loop** integral-differential equation:

$$\boxed{\partial_t \Gamma_k = \frac{1}{2} \,\mathrm{Tr}\, \left(\partial_t R_k \left[\Gamma_k^{(2)} + R_k \right]^{-1} \right)}$$

Wetterich's Flow Equation

with

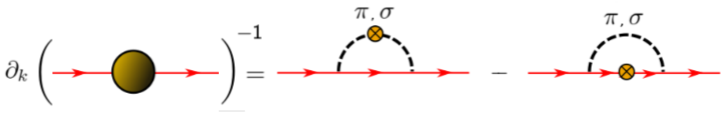
$$t = \ln \left(\frac{k}{\Lambda_{UV}} \right) \qquad \partial_t = k \frac{d}{dk}$$

The properties of this exact flow equation are very convenient for physical calculations since it is an Euclidean **1-loop** integral-differential equation.

Wetterich's equation can be solved applying vertex expansion:

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{p_1 \dots p_n} \Gamma_k^{(n)}(p_1, \dots p_n)\phi(p_1)\dots\phi(p_n)$$

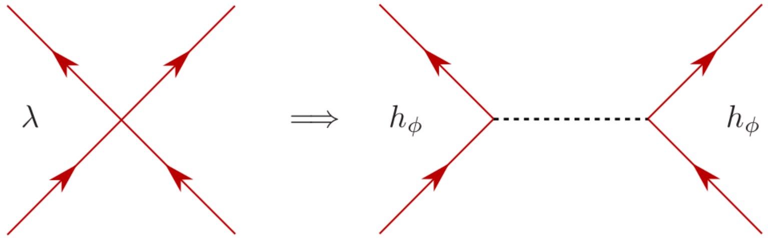
Applying $n-$derivatives and averaging the fields one obtains the flow of the momentum dependent vertex functions.



Remark: a truncation/approximation is needed.

Dynamical Hadronization

Macroscopic QCD degrees of freedom are mesons and baryons. Introduced in the effective action through 4-Fermi Hubbard-Stratonovich transf.:

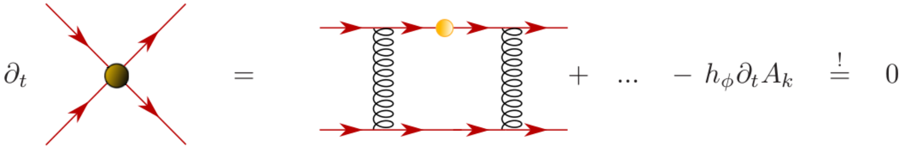


- **Problem**: 4-Fermi interaction flow non-zero, H-S transfomation must be applied in every RG-step $\Longrightarrow$ Solved by Dynamical Hadrnization.

Introduction of scale dependent bosonic field:

$$\partial_t \phi_k(p) = \partial_t A_k(p)(\bar{\psi}\tau\psi)(p) + \partial_t B_k(p)\phi_k(p)$$

with $\partial_t A_k$ and $\partial_t B_k$ defined such that 4-Fermi flow is cancelled:



- This generalizes Hubbard-Stratonovich transf. for every RG-step.
- Green's functions computed with meson exchange diagrams.

Analytical Continuation

Search for bound state properties through real-time Green's functions: analytical continuation must be performed.

Since the inverse propagator of 2-point function is proportional to $(p^2 + M^2)$ in Euclidean Space, goal is to continue to purely imaginary p_0.

- **Extrapolation** by fitting Euclidean momenta p^2 data to a parametrized function and evaluating it at Minkowski momenta $-p^2$.
- **Direct calculation** using the properties of the regulators.

Analytical continuation by extrapolation:

- Padé approximant:

$$R^{(m\,n)}(x) = \frac{\sum_{i=0}^{m} c_i x^i}{1 + \sum_{j=1}^{n} d_j x^j}$$

- Schlessinger point method for a set of M data points:

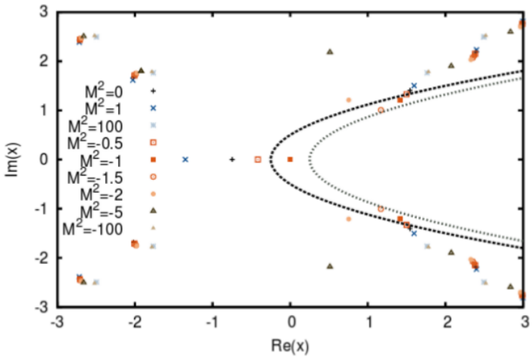
$$C(x) = \cfrac{F(x_1)}{1 + \cfrac{z_1(x-x_1)}{1+\cfrac{z_2(x-x_2)}{\cfrac{\vdots}{z_{M-1}(x-x_{M-1})}}}}$$

Analytical continuation by direct calculation:

- Complex momenta accessed by direct calculation, not extrapolation. Already achieved using 3D-regulators [R-A Tripolt et al, arXiv:1311.0630v2].
- Successfully performed with a 4D modified regulator for zero temperature $O(N)$ model [J. M. Pawlowski and N. Strodthoff, arXiv:1508.01160v3].

$$R_{k;\Delta m_r^2}(p^2) = \left(\Delta\Gamma_k^{(2)}(p^2)|_{\phi=\phi_0} + \Delta m_r^2\right) r\left(\frac{p^2 + \Delta m_r^2}{k^2}\right)$$

- Modified regulator moves poles out the integrating region $\Longrightarrow$ physical implications induced.

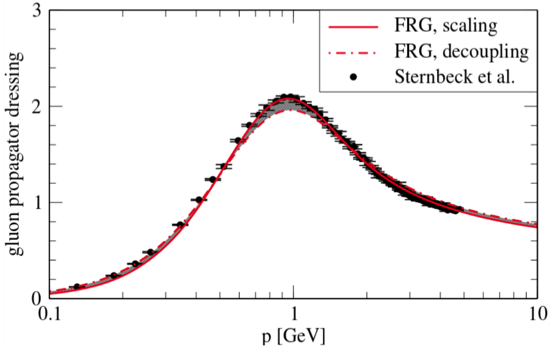


Physical and regulator poles in the complex $x = (p_0^2 + \vec{p}^2)/k^2$ for values of $M^2 = (m^2 - \Delta m_r^2)/k^2$.

[J. M. Pawlowski and N. Strodthoff, arXiv:1508.01160v3]

Preliminary results in the Quark-Meson model

Why using QM as starting point?



A.Cyrol et al, arXiv:1605.01856

Low-energy effective theory with gluons decoupled.

Free massless quark effective action + 4-Fermi interaction:

$$\Gamma_k[\bar{\psi},\psi] = \int \frac{d^4p}{(2\pi)^4} Z_{k,\psi}\, \bar{\psi}_a^A\, i\not{p}\, \psi_a^A \qquad + \qquad \Gamma_k^{4-int}[\bar{\psi},\psi]$$

Applying Hubbard-Stratonovich transformation introducing $\phi = (\sigma, \vec{\pi})$ fields:

$$\begin{aligned}\Gamma_k\left[\bar{\psi},\psi,\sigma,\vec{\pi}\right] &= \int_p \Bigg\{ Z_{\psi,k}\, \bar{\psi}_a^A\, i\not{p}\, \psi_a^A + \\ &+ \frac{1}{2}\left(Z_{k,\phi}p^2 + \overline{m}_{k,\phi}^2\right)\left(\sigma^2 + \pi^z\pi_z\right) - c\sigma + \\ &+ \int_q h_k \bar{\psi}_a^A \left(\frac{\sigma}{2}\delta_{ab} + i\gamma_5(\tau_z)_{ab}\pi^z\right)\psi_b^A \Bigg\}\end{aligned}$$

Results using the LPA'+DH

Wave-function renormalizations $Z_{k,i} \neq 1$ and non-kinetic bosonic terms rewritten in the same O(N) potential:

$$V_k(\rho) = \sum_{n=0}^{\infty} \frac{V_k^{(n)}}{n!} (\rho - \rho_0)^n$$

with $\rho = \frac{1}{2}\left(\sigma^2 + \vec{\pi}^2\right)$ and ρ_0 scale independent expansion point.
For non-zero renormalization wavefunctions the definition of the renormalized couplings/fields is needed:

$$\overline{\rho} = Z_{k,\phi}\rho \qquad \hat{\psi} = Z_{k,\psi}^{\frac{1}{2}}\psi$$

$$\overline{h_k} = \frac{h_k}{Z_{k,\psi} Z_{k,\phi}^{\frac{1}{2}}}$$

Consequences of rewriting effective action:

- Renormalized potential

$$\overline{V_k(\bar{\rho})} = \sum_{n=0}^{\infty} \frac{\overline{V_k^{(n)}}}{n!} (\bar{\rho} - \overline{\rho_0})^n$$

With $\overline{V_k^{(n)}} = \frac{V_k^{(n)}}{Z_{k,\rho}^n}$ and $\overline{\rho_0} = Z_{k,\phi}\rho_0 \Longrightarrow$ running expansion point.

- Terms proportional to (momentum dependent) anomalous dimensions η_i appear explicitly in the flow equation:

$$\eta_{k,i} = -\frac{\partial_t Z_{k,i}}{Z_{k,i}}$$

Two ways to proceed:

- Solving for renormalized couplings.
  - Need to rewrite flow equations with renormalized parameters.
  - Extra terms appear in the flows:

$$\partial_k \overline{h_k} = FLOW + (\eta_\psi + \frac{1}{2}\eta_\rho)\overline{h_k}$$

  - Applied at running $\overline{\rho_0}$.
- Solving for unrenormalized couplings with constant ρ_0 and then translate into renormalized parameters.

Definition of momentum dependent correlators:

$$\Gamma^{(2)}_{k,\phi_i}(p^2) = Z_{k,\phi}(p^2)(p^2 + \overline{m}^2_{k,i}(p^2))$$

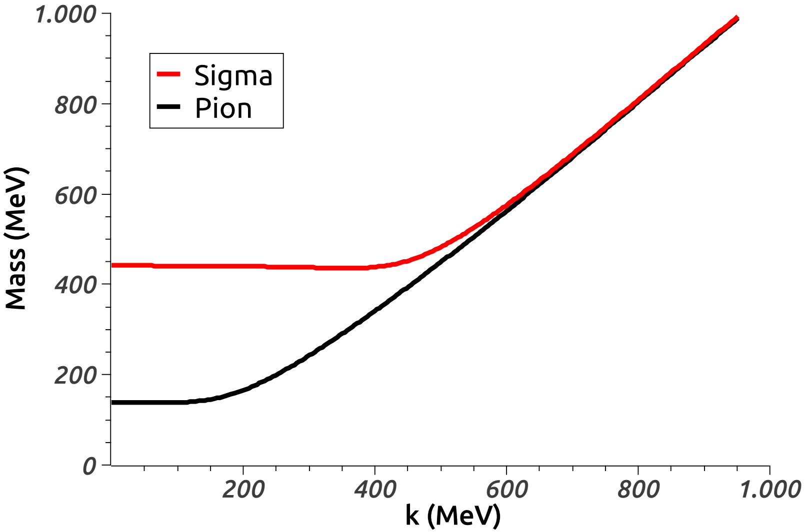
$$\Gamma^{(2)}_{k,\psi}(p^2) = Z_{k,\psi}(p^2)(i\not{p} + \overline{m}^2_{k,\psi}(p^2))$$

Momentum dependence introduced from functional derivative.

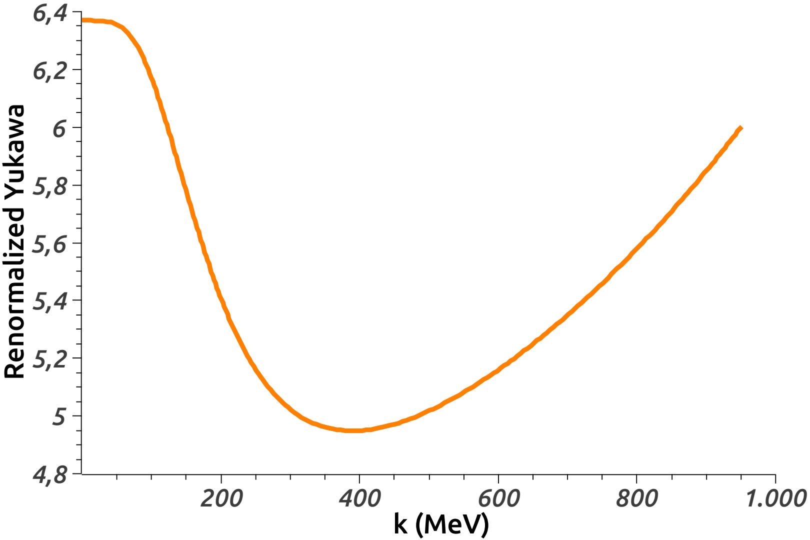
$$\overline{m}^2_{k,\pi}(p^2) = \overline{V^{(1)}_k}(p^2) = \frac{m^2_{k,\pi}(0)}{Z_{k,\phi}(p^2)}$$

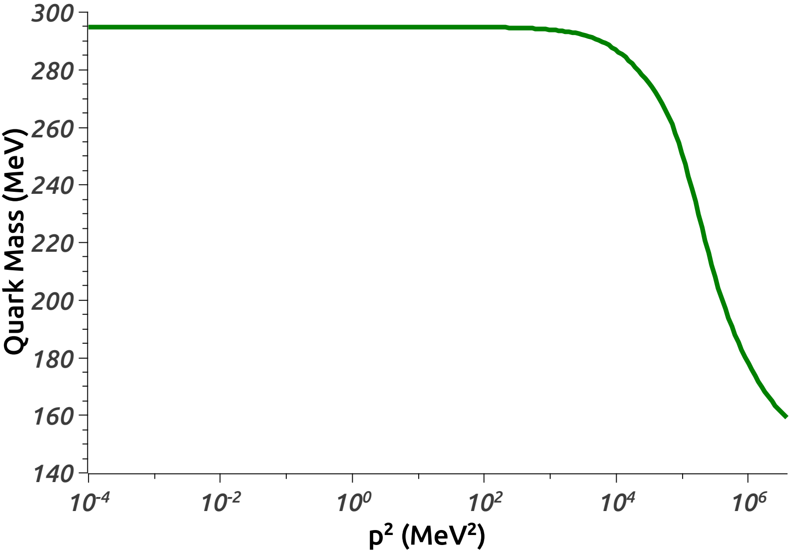
$$\overline{m}^2_{k,\sigma}(p^2) = \overline{V^{(1)}_k}(p^2) + 2\overline{\rho}(p^2)\overline{V^{(2)}_k}(p^2) = \frac{m^2_{k,\sigma}(0)}{Z_{k,\phi}(p^2)}$$

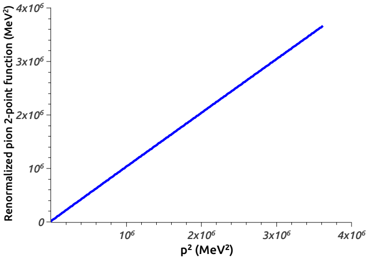
$$\overline{m}_{\psi}(p^2) = \overline{h}_k(p^2)\frac{\overline{\sigma}}{\sqrt{2N_f}}$$

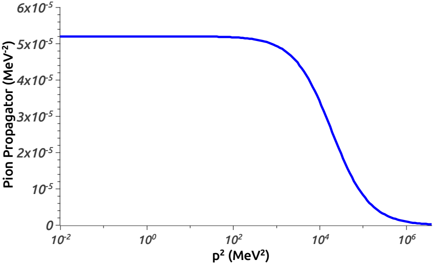


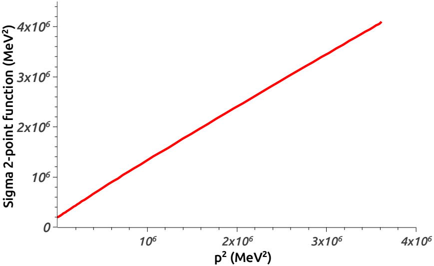
Sigma
Pion
Mass (MeV)
k (MeV)
0
200
400
600
800
1.000
200
400
600
800
1.000

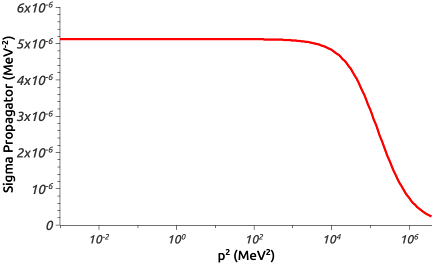

6,4
6,2
6
5,8
5,6
5,4
5,2
5
4,8
Renormalized Yukawa
200
400
600
800
1.000
k (MeV)


300
280
260
240
220
200
180
160
140
Quark Mass (MeV)
10^{-4}
10^{-2}
10^{0}
10^{2}
10^{4}
10^{6}
p^2 (MeV^2)


Renormalized pion 2-point function (MeV²)
4x10⁶
3x10⁶
2x10⁶
10⁶
0
10⁶
2x10⁶
3x10⁶
4x10⁶
p² (MeV²)


Pion Propagator (MeV⁻²)
6x10⁻⁵
5x10⁻⁵
4x10⁻⁵
3x10⁻⁵
2x10⁻⁵
10⁻⁵
0
10⁻²
10⁰
10²
10⁴
10⁶
p² (MeV²)


Sigma 2-point function (MeV²)
4x10⁶
3x10⁶
2x10⁶
10⁶
0
10⁶
2x10⁶
3x10⁶
4x10⁶
p² (MeV²)


Sigma Propagator (MeV⁻²)
6x10⁻⁶
5x10⁻⁶
4x10⁻⁶
3x10⁻⁶
2x10⁻⁶
10⁻⁶
0
10⁻²
10⁰
10²
10⁴
10⁶
p² (MeV²)

Applying analytical continuation and comparing with curvature masses $\overline{m}_{k,\pi}(0) = 138.5$ MeV and $\overline{m}_{k,\sigma}(0) = 441.7$ MeV, we obtained:

- Pole $m_\pi \approx 137.3 \pm 0.4$ MeV
- Pole $m_\sigma \approx 395 \pm 10$ MeV

Results compatible with Schlessinger with the number of points used.

- Pole $m_\pi \approx 137.953$ MeV
- Pole $m_\sigma \approx 382.961$ MeV

Conclusions FRG results:

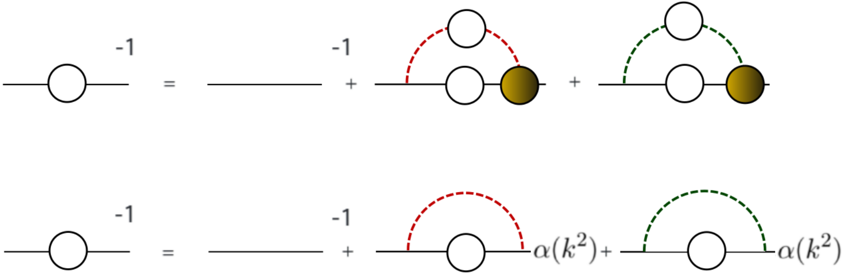
- Calculation of momentum dependent anomalous dimensions and solving the system requires a large numerical effort.
- In terms of pole masses, Padé approximant and Schlessinger point method works fairly well for the pion, but differs for the sigma meson due to the larger mass.
- Schlessinger point method stability requires large number of data points.
- Quark mass decreases with larger momentum but does not reach smaller values in the Quark-Meson model validity range.
- Behaviour agrees in general with QCD results from DSE-BSE at low energies.

Preliminary comparison with the DSE-BSE approach

The good behaviour of the results obtained from the FRG in the LPA'+DH approximation agrees quite well with DSE-BSE results in QCD with the Rainbow-Ladder approximation. Hence, we ought to:

- Reproduce the same results for the **Quark-Meson model in the DSE-BSE approach** using a Rainbow-Ladder-like approximation.
- Establish relation between FRG and DSE-BSE approximations.

Diagrammatic expression for the quark propagator DSE:

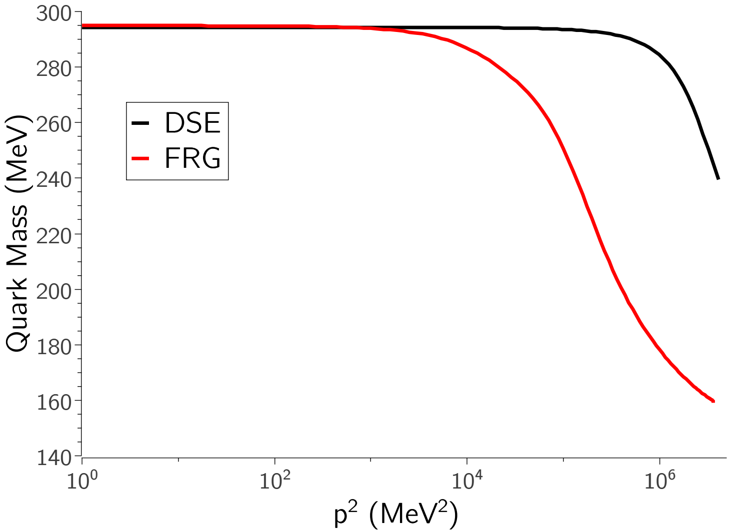


The parameter $\alpha(k^2)$ needs to be modeled. Requiring:

- Cancellation of the meson loop contribution for momenta larger than the validity range of the Quark-Meson model.
- Convergence of the system given the same boundary conditions as in the FRG.

Several models are being tried, for instance:

$$\alpha(k^2) = \left(\frac{k^2}{\eta}\right)^2 \exp\left(-\left(\frac{k^2}{\lambda}\right)^4\right)$$


Quark Mass (MeV)
300
280
260
240
220
200
180
160
140
DSE
FRG
10^0
10^2
10^4
10^6
p^2 (MeV^2)

Summary and Outlook

- The FRG provides an alternative procedure to the BSE/Faddeev equation to obtain resonance masses and decay widths.
- Analytical continuation of the 4-fermi interaction to timelike momenta are to be performed in the Quark-Meson model.
- Big numerical effort and tools are needed to obtain accurate results.
- Systematic comparison of DSE/BSE vs FRG results for QCD.

THANK YOU FOR YOUR ATTENTION