Linear sigma model on the lattice: a learning example

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- Observables
- Renormalization
- Some more observables
- Outlook

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Motivation

- Late 80's, early 90's: (triviality) bounds on the Higgs mass, from MC simulations.
  Many general results are already done (e.g. Huang et al., Phys. Rev. D **35**, 3187 (1987).; Hasenfratz et al., Phys. Lett. B **199** (1987) 531). However, **no MC results on LCP.**

- Recent interest: flux representation combined with worm algorithm can **overcome the sign problem** at finite density. Gattringer and Kloiber, Nucl. Phys. B **869**, 56 (2013); Katz et al., Phys. Rev. D **95**, 054506 (2017).

- Order of the phase transition?
  $\rightarrow$ continuum calculations seem to agree on 2nd order, with few exceptions. But certain truncations do give a first order.
  $\rightarrow$ One outlying MC study: 1st order for small couplings. Bordag et al., Int. J. Mod. Phys. A **27**, 1250116 (2012).

- The ultimate test in scalar models, comparison of 2PI results.

- Understanding triviality from a different perspective. Is there a way to approximately define a continuum limit?

- I want to learn MC simulation techniques ☺

Action

Continuum:

$$S = \int d^4x \frac{1}{2}\left(\partial_\mu\phi_0 \cdot \partial_\mu\phi_0 + m_0^2\phi_0(x)\cdot\phi_0(x)\right) + \frac{g_0}{4!}\left(\phi_0(x)\cdot\phi_0(x)\right)^2 + h(x)\cdot\phi_0(x)\,.$$

Discretised:

$$\begin{aligned} S &= a^4 \sum_x \left\{ \frac{1}{2}\sum_{\hat{\mu}=1}^{4} \left(\phi_0(x+a\hat{\mu}) - \phi_0(x)\right)\cdot\left(\phi_0(x+a\hat{\mu}) - \phi_0(x)\right) \right. \\ &\quad \left. + \frac{m_0^2}{2}\phi_0(x)\cdot\phi_0(x) + \frac{g_0}{4!}\left(\phi_0(x)\cdot\phi_0(x)\right)^2 + h(x)\cdot\phi_0(x) \right\}. \end{aligned}$$

Rewriting in terms of the hopping parameter:

$$\begin{aligned} S &= \sum_x \left\{ \varphi(x)\cdot\varphi(x) - 2\kappa\sum_{\hat{\mu}=1}^{4}\left(\varphi(x)\cdot\varphi(x+a\hat{\mu})\right) + \lambda(\varphi(x)\cdot\varphi(x) - 1)^2 \right. \\ &\quad \left. + H(x)\cdot\varphi(x) - \lambda \right\}, \end{aligned}$$

with $a\phi_0 = \sqrt{2\kappa}\varphi$, $g_0 = 6\lambda/\kappa^2$, $a^3h = H/\sqrt{2\kappa}$ and $a^2m_0^2 = (1-2\lambda)/\kappa - 8$.

Action

- Expressed in terms of bare quantities.
- The renormalized field is defined as $\phi_0 = \sqrt{Z}\phi_R$ and correspondingly $\varphi = \sqrt{Z}\varphi_R$.
- h also renormalizes, since $h \cdot \phi_0 = h_R \cdot \phi_R$, therefore $h_R = Z^{-1/2}h$ and $H_R = Z^{-1/2}H$.

We define the partition function

$$\mathcal{Z}[h] = \mathcal{N}^{-1} \int [d\phi] \, \mathrm{e}^{\,-S[\phi]} \, ,$$

with $\mathcal{N} = \mathcal{Z}[0]$.

Observables - What and how?

The phyiscal quantities we measure:

- Field expectation value,
- Pole masses,
- Thermodynamical quantities:
  $\rightarrow$ (pseudo-)critical temperature,
  $\rightarrow$ pressure,
- Order of the phase transition at $h = 0$.

Observables - What and how?

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The lattice we measure them on:

- Path integrals are carried out using MC simulations. We generate a Markov-chain of ϕ configurations with $P = \mathcal{N}^{-1} \mathrm{e}^{-S[\phi]}$.
- Primary quantities are the averages (noted as $\langle . \rangle$) of operators along the Markov-chain.
- Secondary quantities are functions of primary quantities.
- $N_{\mathrm{S}}^3 \times N_{\mathrm{T}} \equiv N$ lattice sites.
- Periodic boundary conditions.
- Metropolis/heatbath/Hybrid MC algorithm mixed with overrelaxation steps to generate the configurations.

Observables - Field expectation value

- $O(n)$ symmetric case $\leftrightarrow$ φ is a n-element vector (φ^i, $i = 0...n-1$).
- Direction of (constant) $H \leftrightarrow$ 0-th component.
- $i = 0$ is the sigma (σ) direction and the rest are the pion (π) directions.

Definition of the **renormalized** $\bar{\phi}$ is **ambiguous** (we will use notations $\bar{\phi}^0$ and $\bar{\phi}^\sigma$)

$$\bar{\phi}^0 = \sqrt{\frac{2\kappa}{Z}} N^{-1} \sum_x \frac{\varphi(x) \cdot H}{|H|} \equiv \sqrt{\frac{2\kappa}{Z}} N^{-1} \sum_x \varphi^0(x)\,,$$

or, define $\bar{\varphi} = N^{-1} \sum_x \varphi(x)$ and then

$$\bar{\phi}^\sigma = \sqrt{\frac{2\kappa}{Z}} N^{-1} \sum_x \frac{\varphi(x) \cdot \bar{\varphi}}{|\bar{\varphi}|}\,.$$

Hasenfratz et al., Nucl. Phys. B **317**, 81 (1989).

Observables - Field expectation value

- $\langle\bar{\phi}^0\rangle$ coincides with the definition $\dfrac{\delta \log \mathcal{Z}[h]}{\delta h}$.

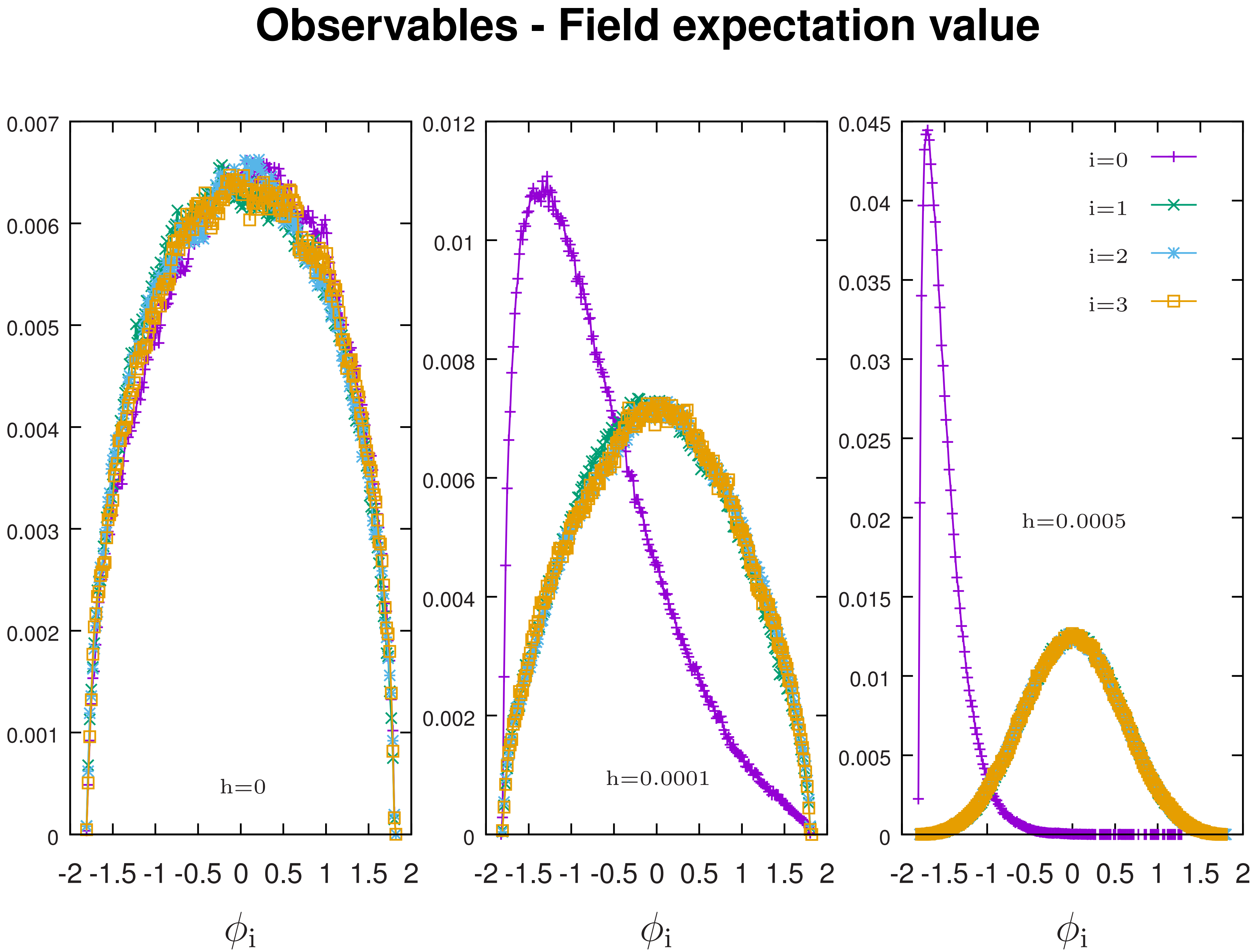
- Due to the finite volume (**no SSB**) $\langle\bar{\phi}^0\rangle \equiv 0$ at vanishing source.

- $h = 0$ value of the expectation value of $\bar{\phi}^0$ can be obtained as a limit.

- $\langle\bar{\phi}^\sigma\rangle$ is non-vanishing even for $h = 0$ in the broken phase.

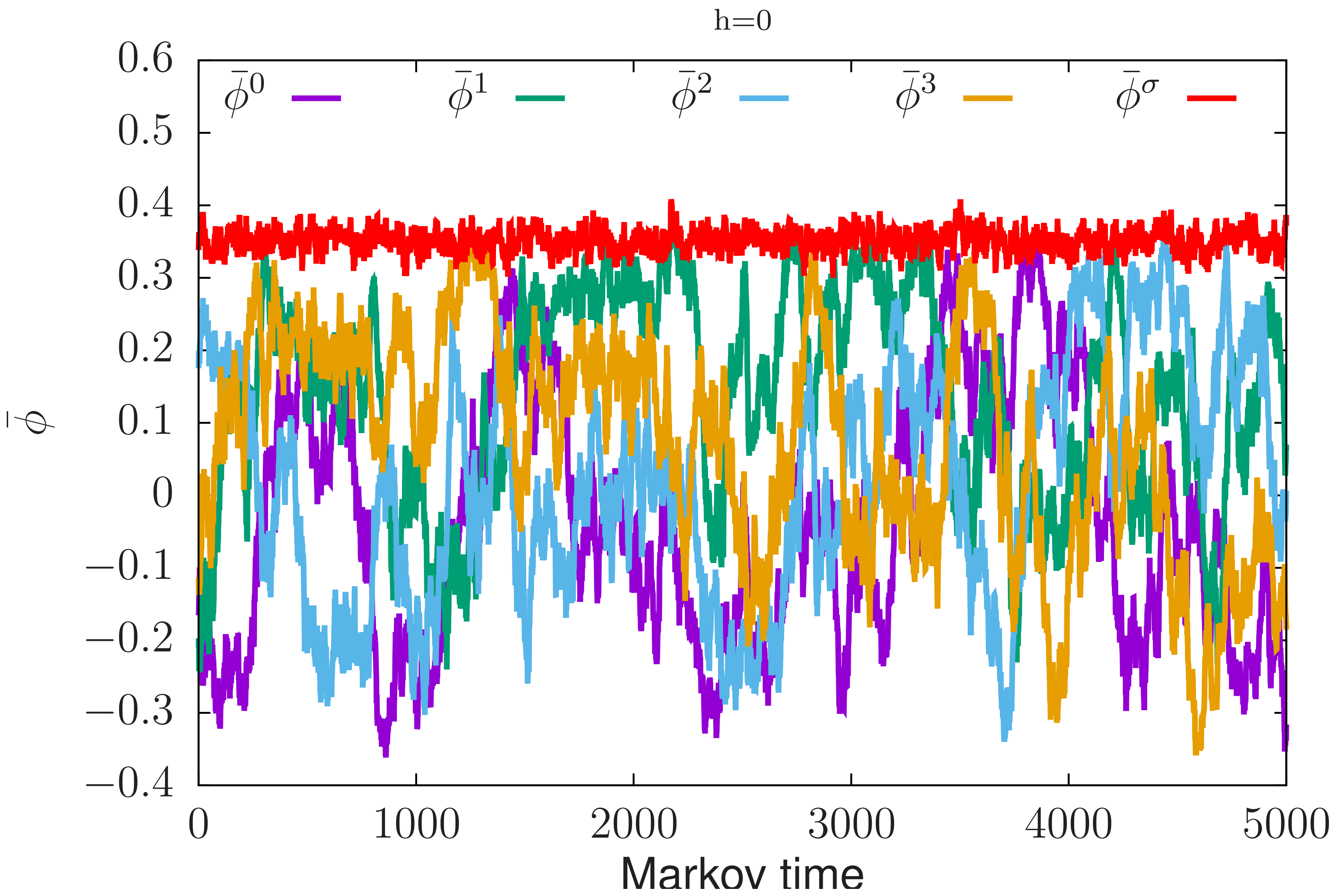
- $\langle\bar{\phi}^\sigma\rangle$ only coincides with $\langle\bar{\phi}^0\rangle$ in the $h \to \infty$ limit.

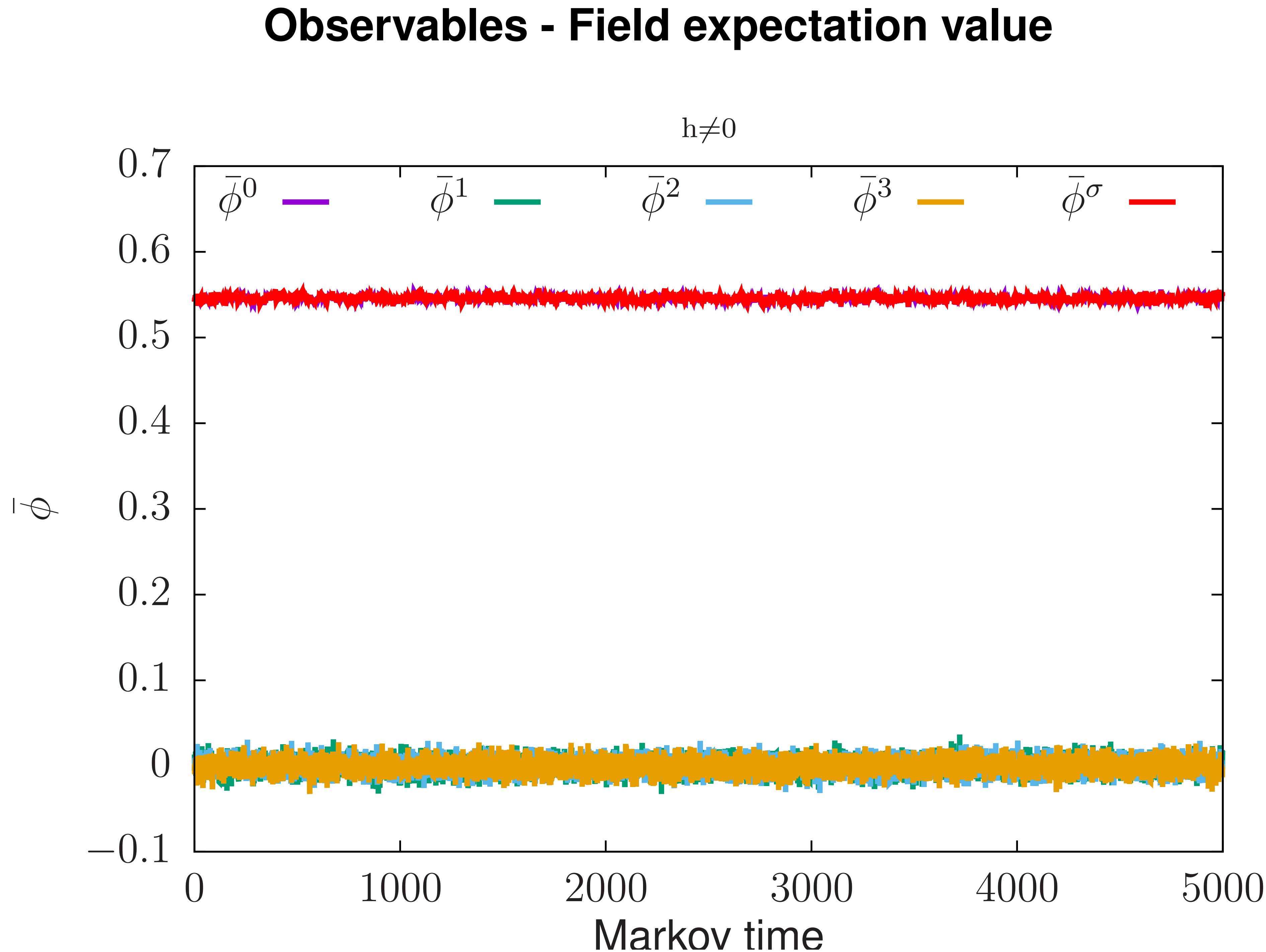
- Whichever definition we choose the **renormalized** vacuum expectation value is the **pion decay constant** (f_π), and can be (in continuum studies usually is) used in the parametrization of the model.



Observables - Field expectation value
h=0
h=0.0001
h=0.0005
i=0
i=1
i=2
i=3
ϕ_i
ϕ_i
ϕ_i

Observables - Field expectation value




Observables - Field expectation value
h≠0
$\bar{\phi}^0$
$\bar{\phi}^1$
$\bar{\phi}^2$
$\bar{\phi}^3$
$\bar{\phi}^\sigma$
$\bar{\phi}$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
0
1000
2000
3000
4000
5000
Markov time

Observables - Pole masses

We define **timeslice** operators as

$$s^i(t) = N_{\mathrm{S}}^{-3} \sum_{x \in \Lambda_t} \varphi^i(x)\,,$$

Λ_t is the sublattice containing all points with time coordinate t.
The time correlator (propagator) matrix

$$C^{ij}(t) = N_{\mathrm{T}}^{-1} \left\langle \sum_{t'} s^i(t+t') s^j(t') \right\rangle .$$

$$\lim_{t \to \infty} C^{ij}(t) \sim \mathrm{e}^{\,-mt}\,,$$

with m the lowest mass in that channel.

At $h \neq 0$ the sigma and pion channels decouple, and the corresponding diagonal elements are dominated by m_σ or m_π respectively.

Observables - Pole masses

The sigma propagator at $h = 0$ can be reached either by extrapolation, or by using timeslice operators constructed similarly to $\bar{\phi}^{\sigma}$

$$s^{\sigma}(t) = N_{\mathrm{S}}^{-3} \sum_{x \in \Lambda_t} \frac{\varphi(x) \cdot \bar{\varphi}}{|\bar{\varphi}|} \,.$$

Then we define the time correlator

$$C^{\sigma}(t) = N_{\mathrm{T}}^{-1} \Big\langle \sum_{t'} s^{\sigma}(t + t') s^{\sigma}(t') \Big\rangle \,.$$

Due to the **periodicity** in t direction the correlator behaves as

$$\sim \left(\mathrm{e}^{-mt} + \mathrm{e}^{-m(N_{\mathrm{T}} - t)} \right),$$

which can be fitted to the data very well, to obtain the pole masses.

Observables - Pole masses

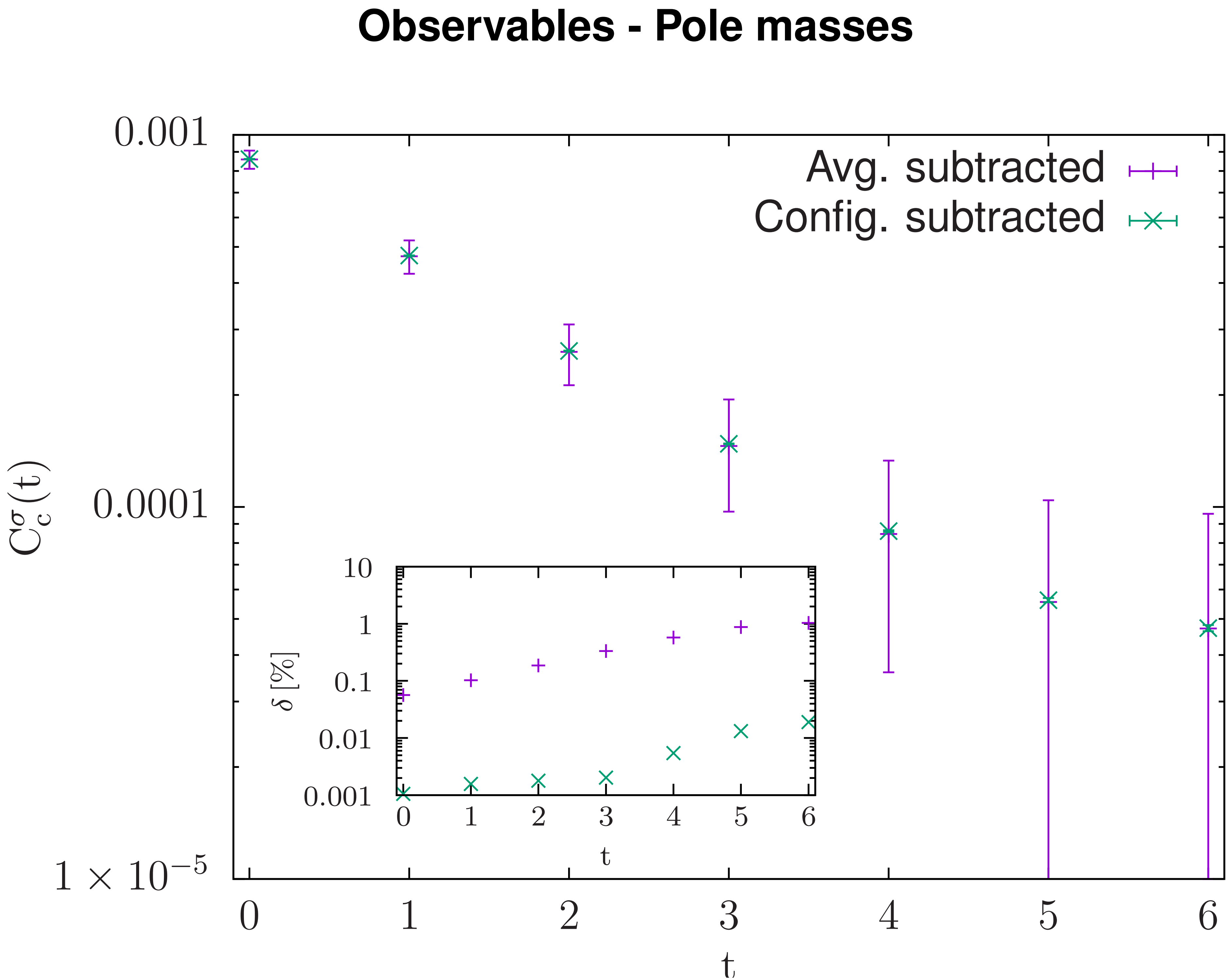
One must **remove the disconnected parts** in the broken phase in order to see that $C(t \to \infty) \to 0$ (up to a constant proportional to $\mathrm{e}^{-mN_\mathrm{S}}$), that is $\langle \bar{\phi}^{0/\sigma} \rangle^2$ respectively. Naïve definition

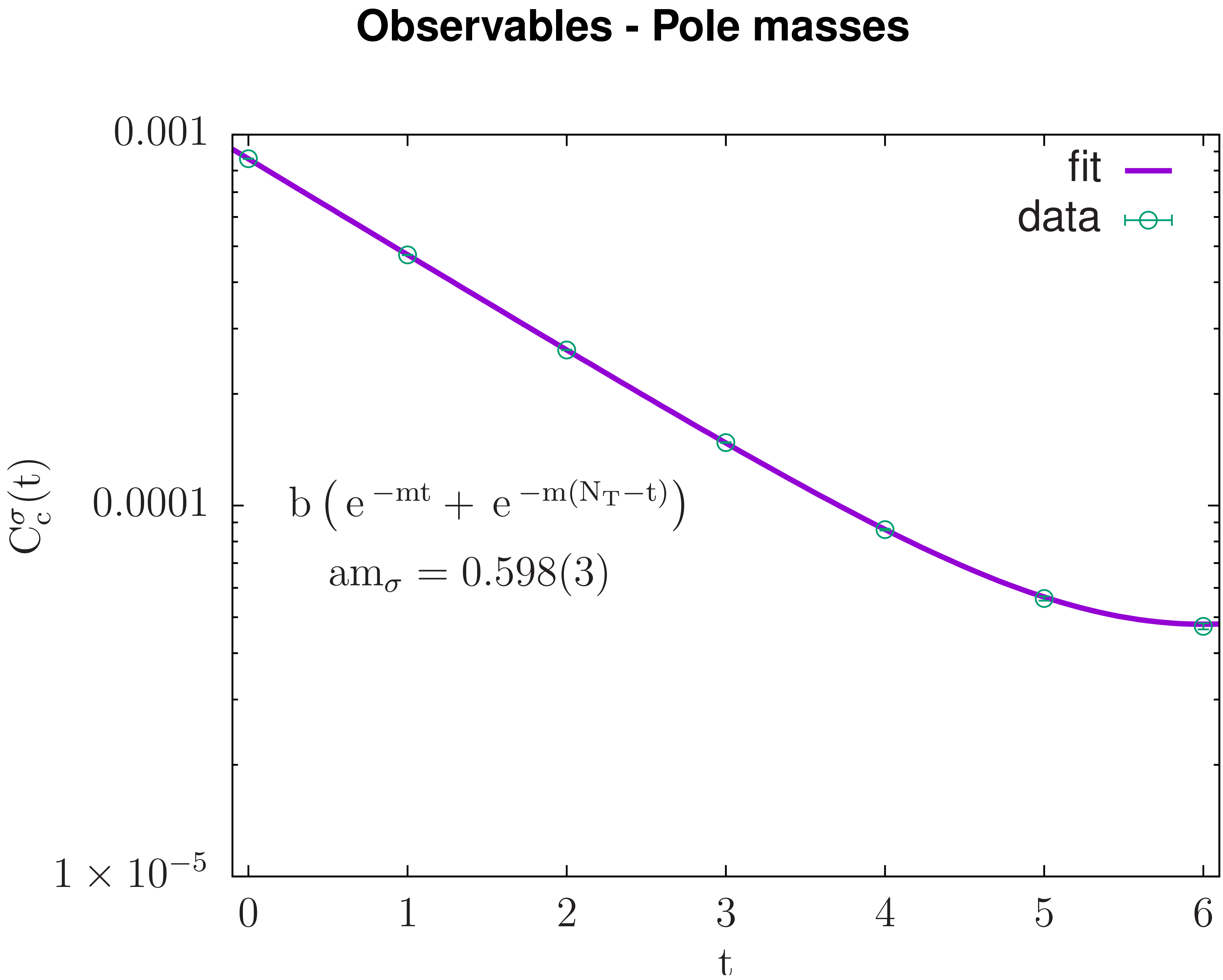
$$C_c^{ij/\sigma} = C^{ij/\sigma}(t) - \langle \bar{\phi}^{i/\sigma} \rangle \langle \bar{\phi}^{j/\sigma} \rangle \,,$$

Interestingly most part of the error of the correlator comes from the disconnected part. Instead of the naïve connected propagator we define the correlator

$$C_c^{ij/\sigma} = N_\mathrm{T}^{-1} \left\langle \sum_{t'} s^{i/\sigma}(t+t') s^{j/\sigma}(t') - \bar{\phi}^{i/\sigma} \bar{\phi}^{j/\sigma} \right\rangle ,$$

which in the infinite volume limit tends to the disconnected propagator (Neuberger, Phys. Rev. Lett. **60**, 889 (1988).).


Observables - Pole masses
Avg. subtracted
Config. subtracted
$C_c^\sigma(t)$
0.001
0.0001
1×10^{-5}
0
1
2
3
4
5
6
t
δ [%]
10
1
0.1
0.01
0.001


Observables - Pole masses
0.001
0.0001
1×10^{-5}
$C_c^{\sigma}(t)$
0
1
2
3
4
5
6
t
fit
data
$b\left(e^{-mt} + e^{-m(N_T - t)}\right)$
$am_{\sigma} = 0.598(3)$

Renormalization:
Line of Constant Physics with the eye of a continuum theorist

- Definition of LCP: a line in the bare parameter space, along which the **ratio** of chosen observables are **constant.**

- We will choose $\bar{\phi}$, m_σ and m_π $\Rightarrow$ **two constraints** in a 3-d space gives a line.

- Along this line the measured **lattice value** of the observables change, that is $a\bar{\phi}$ (or the others respectively). By setting $\bar{\phi}$ to its physical value, $f_\pi = 93$ MeV, we obtain a in **physical units.**

- Moving along the line towards $a \to 0$ we approach the continuum limit.

Renormalization:
Line of Constant Physics with the eye of a continuum theorist

Some thoughts on this process:

- In principle parametrization only forces us to set observables to physical values strictly in the continuum limit.
- Fixing the value of certain observables as functions of a are actually **renormalization conditions.** We are free to choose other conditions.
- Up until the continuum limit different choices give different results.
- This is well exemplified by the fact that we use $\bar{\phi}$ as an observable, which contains the wave function renormalization Z, and therefore can be **redifined by a finite factor.** The choice of Z will be discussed on the next slide.
- While m_σ is uncertain experimentally (altough getting better and better) it seems its physical value is out of the scope of the $O(4)$ model. Nevertheless we want to compare to 2PI in the same model, so we can compare.
- Triviality should appear as a non-analycity along the LCP. An approximate continuum limit is definable if **results scale** with a before the non-analycity affects their behaviour.

Renormalization: what is Z?

The main problem is that only the divergent part of Z is well defined, the finite part is decided by renormalization conditions. There are several definitions in the literature:

- From IR behaviour: $\tilde{C}_c(p=0) \sim Z$, with $\tilde{C}_c$ the Fourier transform of $C_c(t)$.
  Montvay and Munster, "Quantum fields on a lattice".

- From the residue of the propagator around the pole.
  Hasenfratz et al., Phys. Lett. B **199** (1987) 531.

- From Ward identity: $h/\phi = G_\pi^{-1}(p=0)$.
  We require that the r.h.s. $G_\pi^{-1}(p=0) \stackrel{!}{=} Z^{-1} m_\pi^2$, that is $Z = m_\pi^2 \phi / h$.

Observables - Thermodynamics

Pressure: continuum result from 2PI. Fixes the physical value of m_σ.

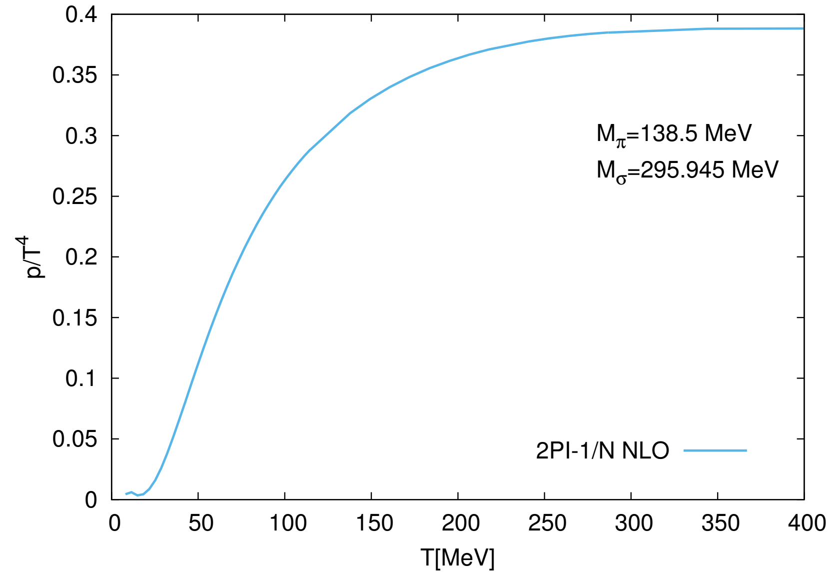
Lattice pressure - ”integral method”:

$$p(T) = \frac{T}{V} \log \mathcal{Z}$$

and

$$\frac{d \log \mathcal{Z}}{d \log a} = \frac{d \log \mathcal{Z}}{d m_0^2} \frac{d m_0^2}{d \log a} + \frac{d \log \mathcal{Z}}{d g_0} \frac{d g_0}{d \log a} + \frac{d \log \mathcal{Z}}{d h} \frac{d h}{d \log a} \, ,$$

which can be integrated back along the LCP to obtain the pressure.



Boyd et al., Nucl. Phys. B **469**, 419 (1996);
Seel et al., JHEP **1307**, 010 (2013).

Pseudo-critical temperature: We define T_{pc} as the maximum of $\frac{d\bar{\phi}(T)}{dT}$. To fine tune the temperature we need to use the LCP and tune $T = aN_T$ either by changing N_T or by chosing different parameters therefore changing a.

Outlook

- Shrink LCP to a line.
- Find the LCP on larger lattices.
- Measure the β-functions on the LCP.
- Using the β-functions, integrate the pressure.
- Find the continuum limit of $T_{\rm pc}$.
- Check the claim that for small bare coupling there is a 1st order PT (a metastable solution exists).

Bordag et al., Int. J. Mod. Phys. A **27**, 1250116 (2012).