Chiral Magnetic Effect with Wigner Functions

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Chiral Magnetic Effect

What is the Chiral Magnetic Effect?

We see a normally unexpected electric current in a non-Abelien system evolving under a strong magnetic field.

Chiral Magnetic Effect

What is the Chiral Magnetic Effect?

- Given a background EM magnetic field, and the QCD gauge fields.
- An initially vanishing chiral imbalance could obtain non-zero value due to the interaction with the gauge fields with non-zero Q_w winding number.

$$Q_w = \frac{g^2}{32\pi^2} \int \mathrm{d}^4x F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a \in \mathbb{Z} \tag{1}$$

Axial charge:

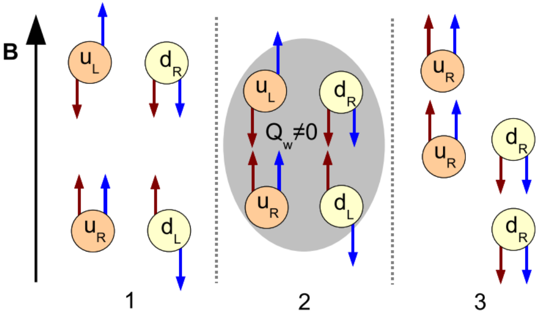
$$(N_L - N_R)_{t=\infty} = 2 N_f Q_w \tag{2}$$

Axial current (on the background field):

$$j^5_\mu = \langle \bar{\psi} \gamma_\mu \gamma_5 \psi \rangle_A \tag{3}$$

D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. A 803, 227 (2008).

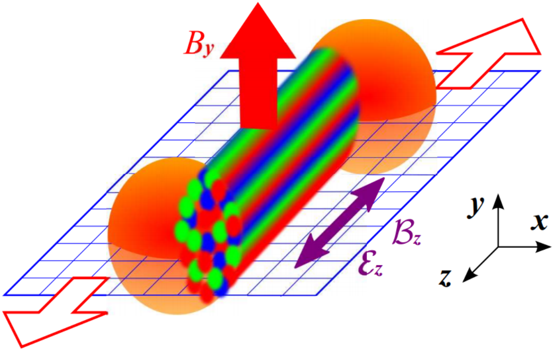
Chiral Magnetic Effect



1. Chirally neutral mixture in very strong B field: particles constrained to Lowest Landau Level.
2. Gauge interaction with non-zero Q_w fields change chirality.
3. Chirality separation leads to charge separation, that leads to current.

Chiral Magnetic Effect

Possible realisation in heavy-ion collisions:



- Background: very strong B field due to highly charged nuclei passing near each other.
- Gauge: QCD gluons

Chiral Magnetic Effect

- Transition between different topologies can happen via tunneling.
- The simplest configuration is a flux-tube, where the gauge fields are $E||B$.
- This can be described by the Schwinger effect $\rightarrow$ connection to pair production.
- Already investigated for constant fields

  Kenji Fukushima, Dmitri E. Kharzeev, and Harmen J. Warringa Phys. Rev. Lett. 104, 212001 2010.
- Main idea: color diagonalisation leads to QED description with E_z, B_z from chromoelectric/magnetic fields and with B_y from EM.

Chiral Magnetic Effect

Main characteristics of the CME (electric) current j_μ:

- $E_z = B_z = B_y = 0$, nothing happens :)
- $E_z = 0, B_z \neq 0, B_y \neq 0$, nothing happens.
- $E_z \neq 0, B_z = 0, B_y = 0$, nothing happens.
- $E_z \neq 0, B_z \neq 0, B_y = 0$, still nothing...
- $E_z \neq 0, B_z = 0, B_y \neq 0$, still nothing...
- Only in the case, when none of the three is zero, is there a CME current!

Chiral Magnetic Effect

- Q: How can we investigate the time dependence of this process?
- A: Generalizing the Schwinger description as usual:
  Wigner functions in the real time formalism.

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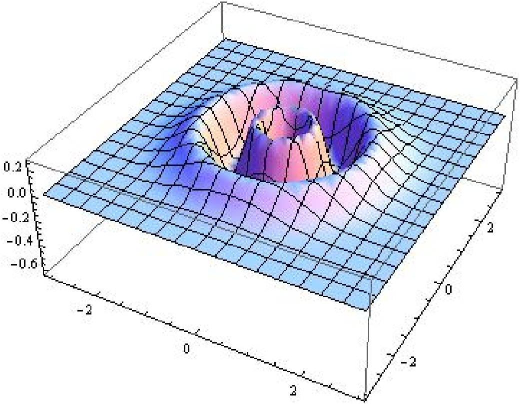
Wigner function

Tool of description: the Wigner function

- Quantum analogue of the classical phase space distribution.



Wigner function of an n=3 Fock state.

Wigner operator

Let's first define the Wigner operator, that is the fourier transform of the density matrix:

$$\hat{\varrho}(x_+, x_-) = \Psi(x_+)\Psi^\dagger(x_-), \quad x_\pm = x \pm \frac{y}{2} \tag{4}$$

where $x = \frac{x_+ + x_-}{2}$ is the center of mass coordinate and $y = x_+ - x_-$ is the relative coordinate, and then

$$\hat{W}(x,p) = \int \frac{\mathrm{d}^4 y}{(2\pi)^4} e^{-ipy} \hat{\varrho}\left(x + \frac{y}{2}, x - \frac{y}{2}\right). \tag{5}$$

Stefan Ochs, Ulrich Heinz, arXiv:hep-th/9806118 1998.

Wigner operator

We can then define the four dimensional Wigner function as the expectation value:

$$W_4(x,p) = \left\langle \hat{W}(x,p) \right\rangle , \tag{6}$$

and its energy average

$$W(\vec{x},\vec{p},t) = \int \mathrm{d}p_0 W(x,p). \tag{7}$$

In contrast to the classical distribution functions, these quantities can be negative at small scales.

Wigner operator

We can recover the classical one particle distribution function by averaging over phase-space areas, that are much larger than the quantum uncertainty scale:

$$f(\vec{x}, \vec{p}, t) = \int_{\Delta V} \frac{\mathrm{d}^3 x' \mathrm{d}^3 p'}{(2\pi)^3} W(\vec{x} - \vec{x}', \vec{p} - \vec{p}', t), \tag{8}$$

with $\Delta V = \Delta^3 x \Delta^3 p \gg (\hbar/2)^3$

Wigner operator

The problem of the 4-dimensional formulation of the Wigner operator, is that it contains 2 time parameters:

$t_1 = x_0 + y_0/2$ and $t_2 = x_0 - y_0/2$.

Now, when fourier transforming we find that

$$W_4(x,p) = \int \frac{\mathrm{d}y_0}{2\pi} e^{-ip_0 y_0} \int \frac{\mathrm{d}^3 y}{(2\pi)^3} e^{-i\vec{p}\vec{y}} \tag{9}$$

$$\left\langle \Psi\left(\vec{x} + \frac{\vec{y}}{2}, x_0 + \frac{y_0}{2}\right) \Psi^\dagger\left(\vec{x} - \frac{\vec{y}}{2}, x_0 - \frac{y_0}{2}\right)\right\rangle, \tag{10}$$

so at any fixed time W_4 depends on $\Psi, \Psi^\dagger$ at all times.

Wigner operator

However, there is no such problem with the single-time, three dimensional Wigner function:

$$W_3(\vec{x},\vec{p},t) = \int \frac{\mathrm{d}^3 y}{(2\pi)^3} e^{-i\vec{p}\vec{y}} \left\langle \Psi(\vec{x}+\frac{\vec{y}}{2},t)\Psi^\dagger(\vec{x}-\frac{\vec{y}}{2},t) \right\rangle . \tag{11}$$

It is also notable, that the single time Wigner function is just the energy integral of the four dimensional Wigner function:

$$W_3(\vec{x},\vec{p},t) = \int \mathrm{d}p_0 W_4\left(x=(t,\vec{x}), p=(p_0,\vec{p})\right) . \tag{12}$$

Wigner operator

Now the compromise is clear:

- The 4 dimensional Wigner function contains all the off-shell physics, that is missing from the single-time Wigner function, but
- The 4 dimensional description cannot be formulated as an initial value problem, and thus it is hard to use in practice.

Wigner operator

One might also consider the higher moments of W_4:

$$W_3{}^{(n)}(\vec{x},\vec{p},t) = \int \mathrm{d}p_0 p_0^n W_4 \left(x = (t,\vec{x}), p = (p_0,\vec{p})\right). \tag{13}$$

It can be shown, that all the information of W_4 is contained in the infinite numer of energy moments $W_3^{(n)}$.

Again, as higher moments has higher time derivates of the fields, the initial value formulation would need all derivates of $\Psi, \Psi^\dagger$ at $t = t_0$ that is the same as knowing them at all times.

The gauge covariant Wigner operator

One might rewrite the density operator definition as follows:

$$\hat{\varrho}\left(x+\frac{y}{2},x-\frac{y}{2}\right)=\Psi(x)e^{-\frac{1}{2}(\partial_x-\partial_x^\dagger)y}\Psi^\dagger(x). \tag{14}$$

If we have gauge fields, we need the Wigner operator to transform accordingly. This can be achieved:

$$\hat{\varrho}\left(x+\frac{y}{2},x-\frac{y}{2}\right)=\bar{\Psi}(x)e^{yD^\dagger(x)/2}\otimes e^{-yD(x)/2}\Psi(x), \tag{15}$$

with the covariant derivate: $D_\mu(x)=\partial_\mu-ig\hat{A}_\mu(x)$.

The gauge covariant Wigner operator

The covariant four dimensional Wigner operator is still the fourier transform:

$$\hat{W}(x,p) = \int \frac{\mathrm{d}^4 y}{(2\pi)^4} e^{-ipy} \bar{\Psi}(x) e^{y D^\dagger(x)/2} \otimes e^{-yD(x)/2} \Psi(x). \tag{16}$$

The covariant derivate makes p the proper kinetic momentum rather than the canonical one.

The equation of motion

We are interested in the time evolution and would like to get an equation of motion for the Wigner operator.

We can insert the Dirac equation and its adjoint into the definition of the Wigner operator and get:

$$
\begin{aligned}
2m\hat{W} &= \gamma^\mu \left(\{\Pi_\mu, \hat{W}\} + i[\Delta_\mu, \hat{W}] \right), \\
2m\hat{W} &= \left(\{\Pi_\mu, \hat{W}\} - i[\Delta_\mu, \hat{W}] \right) \gamma^\mu,
\end{aligned}
\tag{17}
$$

To close the system the evolution of the field strength tensor couples to the current expressed by the Wigner operator:

$$
\left[D^\mu(x), \hat{F}_{\mu\nu}(x) \right] = j_\mu(x) = t_a \, \mathrm{tr} \, \mathrm{t_a} \gamma_\nu \hat{\mathrm{W}}(\mathrm{x}, \mathrm{p}). \tag{18}
$$

The equation of motion

A more intuitive formulation is gained by adding and subtracting the above equations:

$$4m\hat{W} = \left\{\gamma^\mu, \left\{\Pi_\mu, \hat{W}\right\}\right\} + i\left[\gamma^\mu, \left[\Delta_\mu, \hat{W}\right]\right] \quad (19)$$

$$0 = \left[\gamma^\mu, \left\{\Pi_\mu, \hat{W}\right\}\right] + i\left\{\gamma^\mu, \left[\Delta_\mu, \hat{W}\right]\right\}. \quad (20)$$

It can be shown, that the first equation describes generalized mass-shell constraints and the second gives rise to dynamics.

The equation of motion

We have two non-local operators:

$$\Pi_\mu = p_\mu + \frac{2ig\hbar}{c}\int_{-1/2}^{0} \mathrm{d}s\, s\, \partial_p^\nu \hat{F}_{\nu\mu}^{[x]}(x + is\hbar\partial_p) \tag{21}$$

$$\Delta_\mu = \hbar D_\mu(x) - \frac{g\hbar}{c}\int_{-1/2}^{0} \mathrm{d}s\, \partial_p^\nu \hat{F}_{\nu\mu}^{[x]}(x + is\hbar\partial_p) \tag{22}$$

defined in terms of the Schwinger string:

$$\hat{F}_{\nu\mu}^{[x]}(l) = U(x,l)\hat{F}_{\nu\mu}(l)U(l,x), \tag{23}$$

with $l = x + sy$

and the path ordered line element: $U(x,y) = P\exp\left(\frac{ig}{\hbar c}\int_x^y \mathrm{d}z^\mu \hat{A}_\mu(z)\right)$.

The equation of motion

Now, as discussed earlier we need to get to the single time formulation for the energy moments of the covariant four Wigner function.
Formally we end up with a system for the constraints:

$$\hat{W}(\vec{x},\vec{p},t)^{[n+1]} = \sum_{k=0}^{n} O_{k,n}(\ldots)\hat{W}(\vec{x},\vec{p},t)^{[k]} \tag{24}$$

and a similar one for the dynamics:

$$\partial_0 \hat{W}(\vec{x},\vec{p},t)^{[n]} + Q_n(\ldots)\hat{W}(\vec{x},\vec{p},t)^{[n]} = -\sum_{k=0}^{n-1} Q_{k,n}(\ldots)\hat{W}(\vec{x},\vec{p},t)^{[k]} \tag{25}$$

The equation of motion

- It turns out that through Π_0 the nth moment equations couple to the $n+1$th set of equations linking the hierarchy.
- But only the constraint equations contain such dependency, and they have no time derivates.
- The dynamical equation only couples to the $m \leq n$ terms, with the lower order moments playing the role of source, and accordingly for $n = 0$ there is no source term.
- It is possible to derive from these a new infinite set of constraints that can be used to construct all higher moment equations from the $n = 0$ one.

The equation of motion

However, we need to use the following approximations:

- Hartree approximation to distribute the expectation value over operator products: $\langle\hat\rho\hat F_{\mu\nu}\rangle = \langle\hat\rho\rangle F_{\mu\nu}$
- External field (classicality)

Also, we'll specialize to QED.

The equation of motion

The non local operators simplify to:

$$\Pi_\mu \to P_\mu = p_\mu + ie\int_{-1/2}^{1/2} \mathrm{d}s\, s\, F_{\mu\nu}(x + is\partial_p)\partial_p^\nu \tag{26}$$

$$\Delta_\mu \to D_\mu = \partial_\mu - e\int_{-1/2}^{1/2} \mathrm{d}s\, F_{\mu\nu}(x + is\partial_p)\partial_p^\nu \tag{27}$$

And the 0th order equation of motion simplifies to:

$$D_0 W = -\frac{1}{2}\vec{D}[\gamma^0\vec{\gamma}, W] - im[\gamma^0, W] - i\vec{P}\{\gamma^0\vec{\gamma}, W\} \tag{28}$$

The equation of motion

The Wigner function is usually decomposed into real functions by using the Dirac matrix basis:

$$W(\vec{x},\vec{p},t) = \frac{1}{4}\left[\mathbb{1}\mathbb{s} + i\gamma_5\mathbb{p} + \gamma^\mu\mathbb{v}_\mu + \gamma^\mu\gamma_5\mathbb{a}_\mu + \sigma^{\mu\nu}\mathbb{t}_{\mu\nu}\right] \tag{29}$$

Equations of motion for the spin-1/2 Wigner function

We arrive at a system for 16 unknown real functions:

$$
\begin{aligned}
&D_t\mathbb{s} & & & - \quad & 2\vec{P}\cdot\vec{\mathbb{t}}_1 &&= 0 && (30)\\
&D_t\mathbb{p} & & & + \quad & 2\vec{P}\cdot\vec{\mathbb{t}}_2 &&= 2m\mathbb{a}_0 && (31)\\
&D_t\mathbb{v}_0 & + \quad & \vec{D}_{\vec{x}}\cdot\vec{\mathbb{v}} & & &&= 0 && (32)\\
&D_t\mathbb{a}_0 & + \quad & \vec{D}_{\vec{x}}\cdot\vec{\mathbb{a}} & & &&= 2m\mathbb{p} && (33)\\
&D_t\vec{\mathbb{v}} & + \quad & \vec{D}_{\vec{x}}\mathbb{v}_0 & + \quad & 2\vec{P}\times\vec{\mathbb{a}} &&= -2m\vec{\mathbb{t}}_1 && (34)\\
&D_t\vec{\mathbb{a}} & + \quad & \vec{D}_{\vec{x}}\mathbb{a}_0 & + \quad & 2\vec{P}\times\vec{\mathbb{v}} &&= 0 && (35)\\
&D_t\vec{\mathbb{t}}_1 & + \quad & \vec{D}_{\vec{x}}\times\vec{\mathbb{t}}_2 & + \quad & 2\vec{P}\mathbb{s} &&= 2m\vec{\mathbb{v}} && (36)\\
&D_t\vec{\mathbb{t}}_2 & - \quad & \vec{D}_{\vec{x}}\times\vec{\mathbb{t}}_1 & - \quad & 2\vec{P}\mathbb{p} &&= 0 && (37)
\end{aligned}
$$

Simplification

For the CME calculation we are first interested in light quarks...
Let's simplify things: $m = 0$.

Only the vector current / charge and the axial current / charge remains in the equations...

Equations of motion for the $m = 0$ spin-1/2 Wigner function

A system for 8 unknown real functions remains:

$$
\begin{aligned}
D_t \mathbb{v}_0 &+ \vec{D}_{\vec{x}} \cdot \vec{\mathbb{v}} && &&= 0 \quad (38)\\
D_t \mathbb{a}_0 &+ \vec{D}_{\vec{x}} \cdot \vec{\mathbb{a}} && &&= 0 \quad (39)\\
D_t \vec{\mathbb{v}} &+ \vec{D}_{\vec{x}} \mathbb{v}_0 &&+ 2\vec{P} \times \vec{\mathbb{a}} &&= 0 \quad (40)\\
D_t \vec{\mathbb{a}} &+ \vec{D}_{\vec{x}} \mathbb{a}_0 &&+ 2\vec{P} \times \vec{\mathbb{v}} &&= 0 \quad (41)
\end{aligned}
$$

Wigner function non-local operators

Also, to reduce dimensions, we consider homogeneous external fields.
In this case, the non-local operators are given exactly (without gradient expansion) as:

$$D_t = \partial_t + g\vec{\mathcal{E}}(t)\vec{\nabla}_{\vec{p}} \tag{42}$$

$$\vec{D}_{\vec{x}} = g\vec{\mathcal{B}}(t) \times \vec{\nabla}_{\vec{p}} \tag{43}$$

$$\vec{P} = \vec{p} \tag{44}$$

Observables

For the CME calculation we record the time evolution of the phase-space integrals of the currents and charges:

$$\mathbb{v}^{\mu}(t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \mathrm{d}p^3 \mathbb{v}^{\mu}(t, \vec{p}) \ , \tag{45}$$

$$\mathbb{a}^{\mu}(t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \mathrm{d}p^3 \mathbb{a}^{\mu}(t, \vec{p}) \ . \tag{46}$$

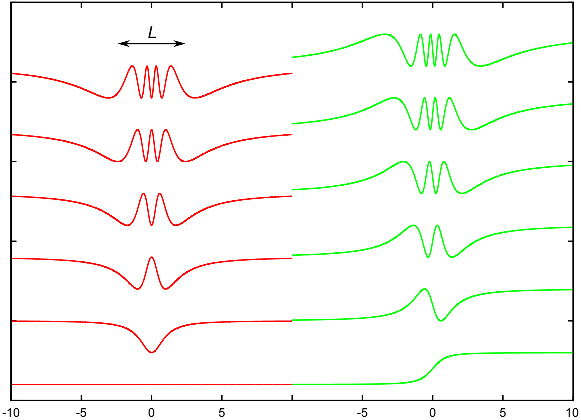
Ingredients of the 3+1 D numerical solver:

- Pseudospectral collocation
- Rational Chebyshev polynomial basis
- 4th order Runge-Kutta
- GPU Acceleration

Numerical Solution

Rational Chebyshev polynomial basis:

$$\mathrm{TB}_n(x) = \cos\left(n \cdot \mathrm{acot}(x/L)\right) \tag{47}$$



Numerical Solution

Rational Chebyshev polynomial basis:

$$\mathrm{TB}_n(x) = \cos\left(n \cdot \mathrm{acot}(x/L)\right) \tag{48}$$

Collocation points:

$$x_i = L \cdot \cot\left(\frac{\pi i}{n+1}\right), \quad i = 1...n \tag{49}$$

Quadrature weights:

$$w_i = \frac{L\pi}{n+1}\frac{1}{\sin^2\left(\frac{i\pi}{n+1}\right)}, \quad i = 1...n \tag{50}$$

Verification

The Wigner function is related to the Feynman propagator:

$$W(\vec{x}, \vec{p}, t) = \frac{i}{2} \int \mathrm{d}^3 s e^{-i\vec{p}\vec{s}} G(\vec{x} + \vec{s}/2, t, \vec{x} - \vec{s}/2, t) \tag{51}$$

That makes possible to derive an exact solution for the static magnetic field case (Bialinicky-Birula, Górnicki, Rafelski).

Also, there exist an analytical solution for the asymptotic energy density $f(t \to \infty)$ for $E(t) = E_0 \cosh^{-2}(t/\tau)$ field (Narozhny, Nikishov).

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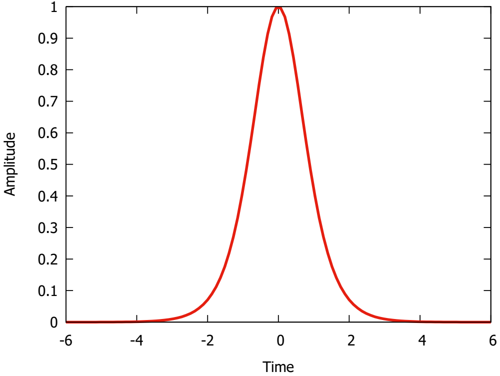
Sauter field

Start with a toy field that we know very well:

$$S(t) = \cosh^{-2}(t/\tau) \tag{52}$$



Sauter field

$$S(t) = \cosh^{-2}(t/\tau) \quad (53)$$

Let the fields be:

$E_z = A \cdot S(t),$

$B_z = A \cdot \cos(\alpha) S(t),$

$B_y = A \cdot \sin(\alpha) S(t)$

Sauter field

Let's check the CME characteristics:

- $E_z = B_z = B_y = 0$, nothing happens :)

This is trivial: without fields, all derivatives vanish in the DHW equations

Sauter field

Let's check the CME characteristics:

- $E_z = 0, B_z \neq 0, B_y \neq 0$, nothing happens.

Magnetic fields alone does not produce electric current.

Sauter field

Let's check the CME characteristics:

- $E_z \neq 0, B_z = 0, B_y = 0$

There is current, but only in the direction of the E field. Nothing interesting.

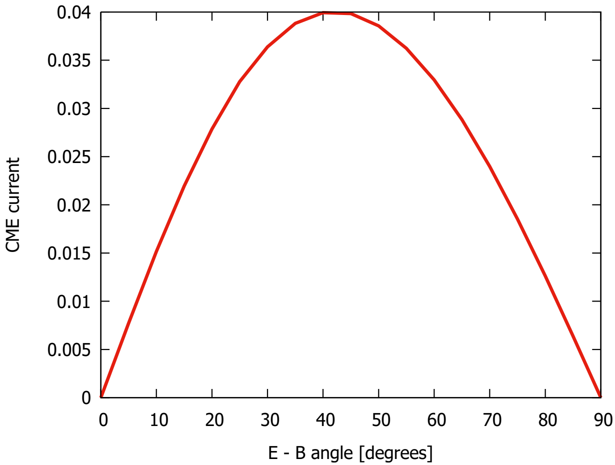
Sauter field

Let's check the CME characteristics:

- $E_z \neq 0, B_z \neq 0, B_y = 0$, still no CME current
- $E_z \neq 0, B_z = 0, B_y \neq 0$, still no CME current
- Only in the case, when none of the three is zero, is there a CME current!

Let's see this!

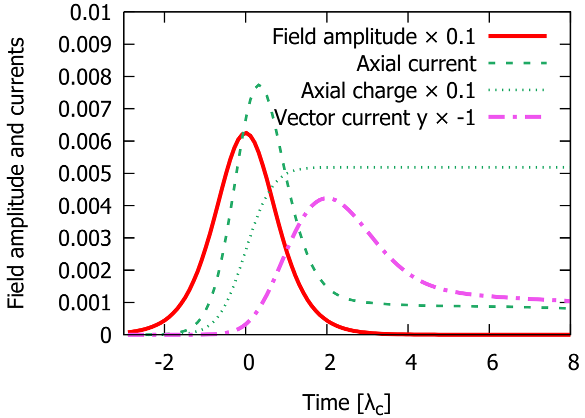
Sauter field



Only in the case, when none of the three is zero, is there a CME current!

Sauter field

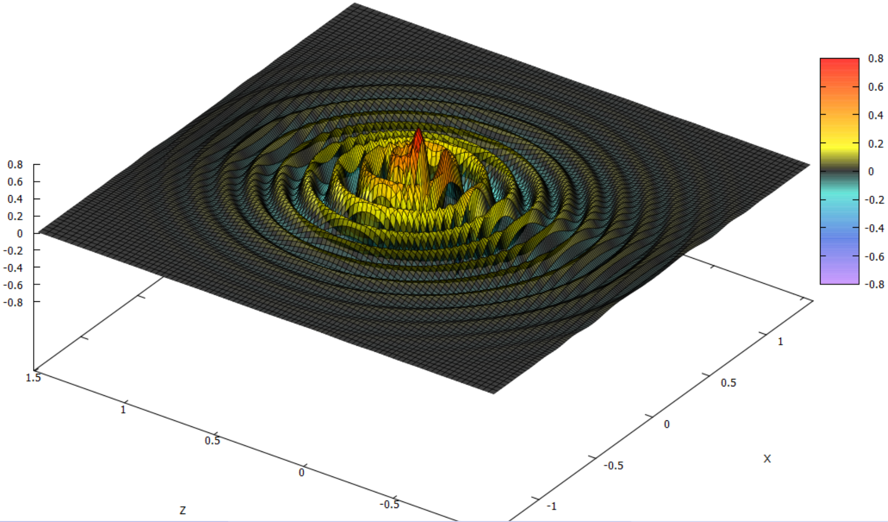
CME formation during the interaction:



Field comes first, then axial current and then electric current!

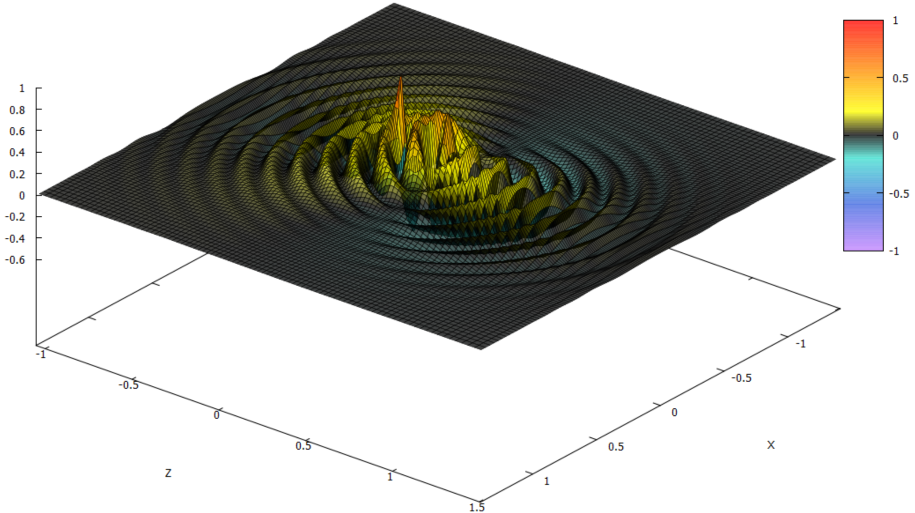
Results

The Anomalous component of the electric current:



Results

The Axial current in X direction:



Toy model field for high energy collisions

Assume the following fields for a high energy collision:
Define the following time dependent function :

$$\Phi(t, \tau, A, \kappa) = A \cdot \begin{cases} \cosh^{-2}(10\mathrm{t}/\tau) & t < 0, \\ (1 + t/\tau)^{-\kappa} & t \geq 0, \end{cases} \tag{54}$$

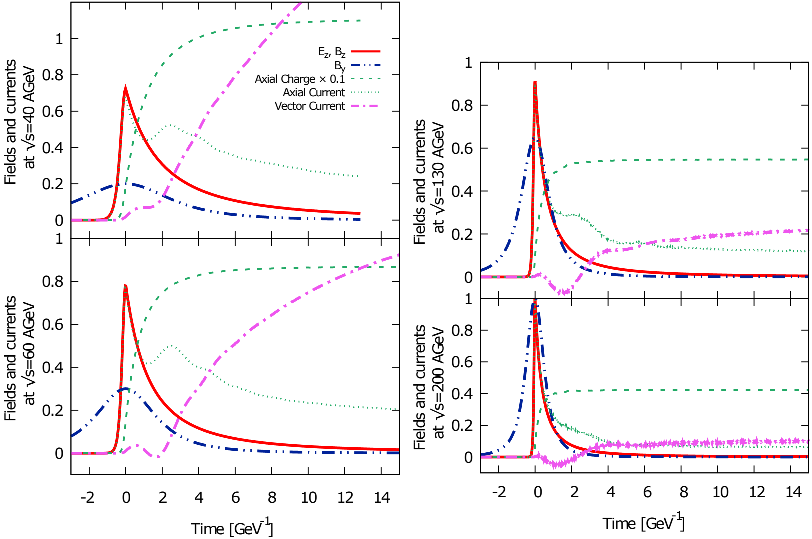
where we set $\kappa = 2$ and model the external fields as:

$$e\vec{E}(t) = \{0, 0, \Phi(t, \tau, A_{Ez}, \kappa)\} , \tag{55}$$

$$e\vec{B}(t) = \{0, A_{By}\left(1 + \frac{t^2}{\tau^2}\right)^{-3/2}, \Phi(t, \tau, A_{Bz}, \kappa)\} . \tag{56}$$

with some phenomenological parameters.

Toy model field for high energy collisions



Summary

- The DHW equations can predict electric and axial currents consistent with what would be expected in the Chiral Magnetic Effect.
- The fields are connected to the electromagnetic B, and the non-Abelian chromoelectromagnetic $\mathcal{E}, \mathcal{B}$ fields.
- The DHW description can act as a real time microscopic source for the effect in non-central heavy-ion collisions
- A phenomenologically parametrized model field shows the disappearance of the effect at high energies and a sign change at intermediate energies.

arxiv:1707.03621

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Wigner GPU Lab