# Minkowski space calculation of the pion form factors

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Non-perturbative methods of Quantum Field Theory
September 20-22, 2017, Zalakaros

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Hrvatska zaklada za znanost

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# Motivations

Example: Separable approximation

$$g^2 G^{\mu\nu} = g^{\mu\nu} D(-k^2)$$

$$D(-(p-k)^2) \approx D_0 f_0(-p^2) f_0(-k^2) - D_1 (p \cdot k) f_1(-p^2) f_1(-k^2)$$

Schwinger–Dyson equation $\Rightarrow$

$$\begin{aligned} A(p^2) &= 1 + a f_1(-p^2) \\ B(p^2) &= m + b f_0(-p^2) \end{aligned}$$

Typical Ansatz [1]:

$$\begin{aligned} f_0(x) &= e^{-x/\Lambda_0^2} \\ f_1(x) &= \frac{1 + e^{-x_0/\Lambda_1^2}}{1 + e^{-(x-x_0)/\Lambda_1^2}} \end{aligned}$$

# Motivations

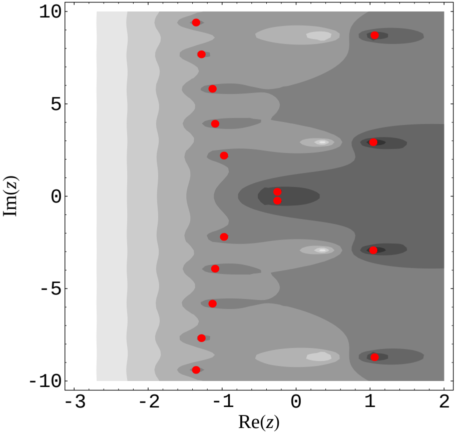
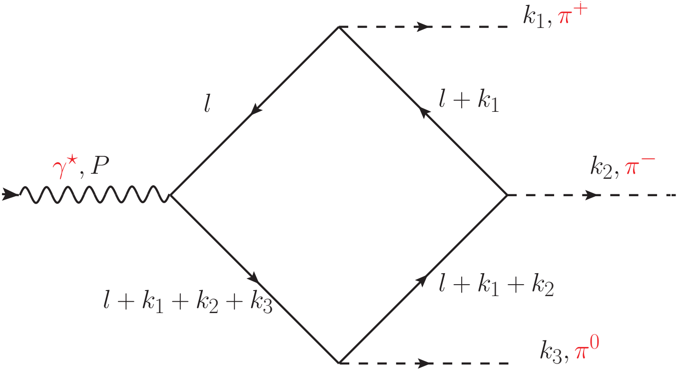



Figure: Model E: Contour plot of the function $z \mapsto \log |A(z)^2 z + B(z)^2|$. The red points are solutions of the equation $A(z)^2 z + B(z)^2 = 0$.

# Motivations

$\gamma^\star \to 3\pi$, just an example




The $\gamma^\star \to 3\pi$ form factor $F^{3\pi}$ is defined as

$$
\begin{aligned}
& e\langle 0|J^\mu(0)|\pi^+(\boldsymbol{p}_+)\pi^-(\boldsymbol{p}_-)\pi^0(\boldsymbol{p}_0)\rangle \\
= \quad & iF^{3\pi}(s_+, s_-, s_0)\, \varepsilon^{\mu\alpha\beta\lambda}\, (p_0)_\alpha\, (p_+)_\beta\, (p_-)_\lambda \ .
\end{aligned}
$$

# Quark Propagator

Parametrization of the quark propagator in covariant gauges:

$$\begin{aligned} S(q) &= -\sigma_V(-q^2)\not{q} - \sigma_S(-q^2) \\ &= Z(-q^2)\frac{\not{q} + M(-q^2)}{q^2 - M^2(-q^2)} \\ &= \frac{A(-q^2)\not{q} + B(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)} \end{aligned}$$

# Quark Propagator of Mello, Melo, and Frederico

[2] Mello, Melo, and Frederico, Phys. Lett. **B766**, 66 (2017)

$$M(x) = (m_0 - i\varepsilon) - m^3 \left[x - \lambda^2 + i\varepsilon\right]^{-1}$$
$$Z(x) = 1$$

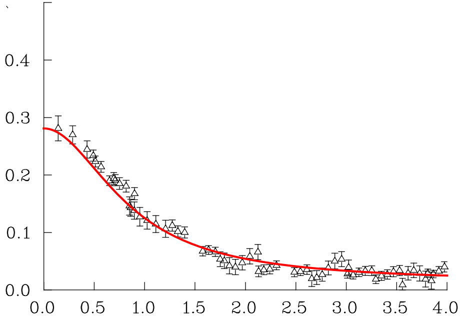
Model parameters: $m_0 = 0.014\ \mathrm{GeV}$, $m = 0.574\ \mathrm{GeV}$, and $\lambda = 0.846\ \mathrm{GeV}$

Asymptotic expansions about $\infty$ and 0:

$$M(x) = m_0 + \frac{m^3}{x} - \frac{\lambda^2 m^3}{x^2} + \mathcal{O}((\frac{1}{x})^3) , \qquad \text{for } x \to \infty ,$$
$$M(x) = \left(m_0 + \frac{m^3}{\lambda^2}\right) - \frac{m^3 x}{\lambda^4} + \frac{m^3 x^2}{\lambda^6} + \mathcal{O}(x^3) , \qquad \text{for } x \to 0 ,$$

# Quark Propagator of Mello, Melo, and Frederico

Lattice data: [3] Parappilly *et al.*, Phys. Rev. D **73**, 054504 (2006).




$M(q)\,[\mathrm{GeV}]$ vs. $q\,[\mathrm{GeV}]$

# Quark Propagator of Mello, Melo, and Frederico

The quark dressing functions $\sigma_V$ and $\sigma_S$:

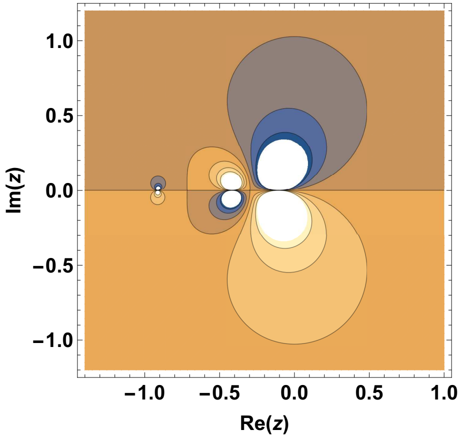
$$\sigma_V(x) = \sum_{j=1}^{3} \frac{b_{Vj}}{x + a_j}$$

$$\sigma_S(x) = \sum_{j=1}^{3} \frac{b_{Sj}}{x + a_j}$$

The mass parameters:

$$a_1 = 0.1046 \text{ GeV}^2$$
$$a_2 = 0.4160 \text{ GeV}^2$$
$$a_3 = 0.9110 \text{ GeV}^2$$

# Quark Propagator of Mello, Melo, and Frederico




Contour plot of Im($\sigma_V(z)$) in the complex $z$-plane.

# 3R Quark Propagator

[4] Alkofer, Detmold, Fischer, and Maris, Phys.Rev. D **70** 014014, (2004)
The dressing functions $\sigma$ are:

$$\sigma_V(x) = \frac{1}{Z_2}\sum_{j=1}^{3}\frac{2r_j(x+a_j^2-b_j^2)}{(x+a_j^2-b_j^2)^2+4a_j^2b_j^2}$$

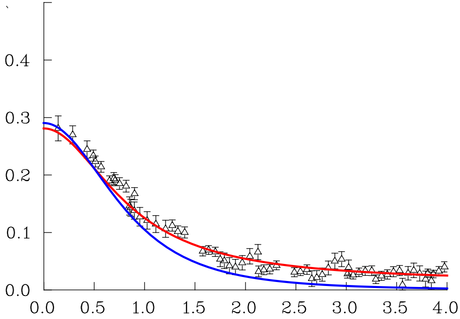
$$\sigma_S(x) = \frac{1}{Z_2}\sum_{j=1}^{3}\frac{2r_ja_j(x+a_j^2+b_j^2)}{(x+a_j^2-b_j^2)^2+4a_j^2b_j^2}$$

Quark propagator:

$$S(q) = -\sigma_V(-q^2)\not{q} - \sigma_S(-q^2) = \sum_{j=1}^{3}\frac{A_j\not{q}+B_j}{A_j^2q^2-B_j^2}$$

# 3R Quark Propagator

Lattice data: [3] Parappilly *et al.*, Phys. Rev. D **73**, 054504 (2006).




$M(q)$ [GeV] vs. $q$ [GeV]

# Pion Decay Constant

$$f_\pi = i\frac{N_c}{2}\frac{1}{M_\pi^2}\int\frac{d^4q}{(2\pi)^4}\mathrm{tr}\left(\not{P}\gamma_5 S(q+\frac{P}{2})\left(-\frac{2B(-q^2)}{f_\pi}\gamma_5\right)S(q-\frac{P}{2})\right)$$

Mello: $B(x) = M(x)|_{m0=0} = -m^3\left[x-\lambda^2+i\varepsilon\right]^{-1}$

(1) Calculation using FeynCalc [5, 6] and Package-X [7, 8] (or LoopTools [9])

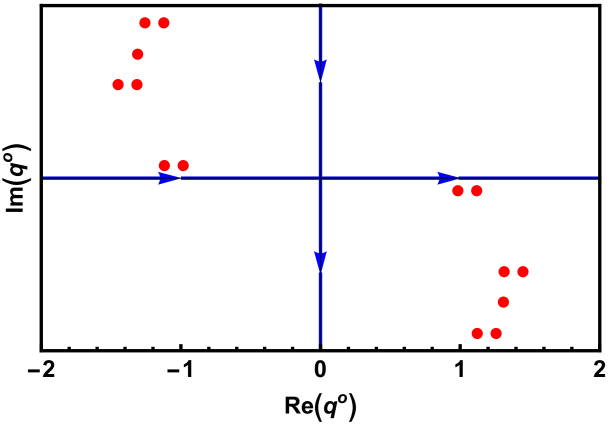
$$f_\pi = 87.5599\ \mathrm{MeV}$$

(2) Euclidean integration

- "naïve" Wick rotation ($q^0 \to -iq^0$) is correct here
- two nontrivial integration (red variables) $q = \left(q^0, \xi\sin(\vartheta)\cos(\varphi), \xi\sin(\vartheta)\sin(\varphi), \xi\cos(\vartheta)\right)$
- numerical integration using Mathematica
- the same result for $f_\pi$

# Pion Decay Constant

Poles of the integrand ($\xi = 1$):



# Pion Decay Constant

(3) Minkowski space integration utilizing light-cone momenta

- This calculation follows Mello, Melo, and Frederico paper [2]
- light–cone momenta $q_{\pm} = q^0 \pm q^3$
- seven simple poles on the real $q_{-}$ axis
  - Cauchy's residue theorem is used
  - residua are calculated analytically
  - remaining integrals over $q_{+} \in [-\frac{M_\pi}{2}, \frac{M_\pi}{2}]$ and $(q_1)^2 + (q_2)^2$ are calculated numerically
- the same result for $f_\pi$

Mello, Melo, and Frederico [2]: $f_\pi = 90\ \mathrm{MeV}$

# Pion Decay Constant

Calculation using 3R Quark Propagator [4]:

(1) Euclidean or Minkowski space calculation

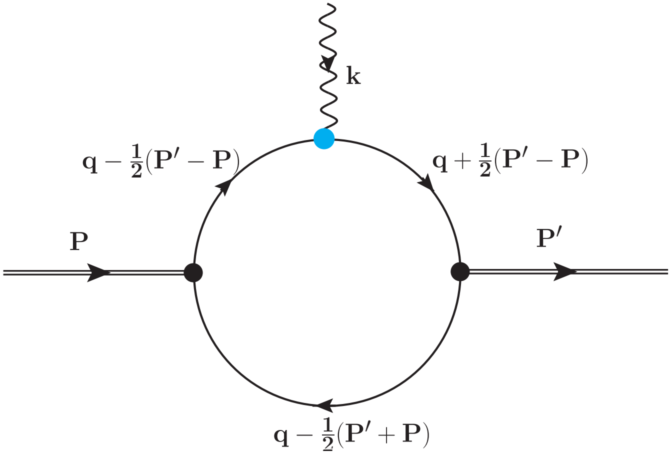
$$f_\pi = 71.5611 \, \mathrm{MeV}$$

- Euclidean integration: two nontrivial integration
- Minkowski integration: only residua contribute, principal part = 0

(2) Calculation using FeynCalc and LoopTools

$$f_\pi = 71.5614 \, \mathrm{MeV}$$

# Electromagnetic form factor



# Electromagnetic form factor

Matrix element:

$$
\begin{aligned}
&\langle \pi^+(P')|J^\mu(0)|\pi^+(P)\rangle \\
&= (P^\mu + P'^\mu)F_\pi(Q^2) \\
&= i(Q_u - Q_d)\frac{N_c}{2}\int \frac{d^4q}{(2\pi)^4}\mathrm{tr}\Big\{\bar{\Gamma}(q-\frac{P}{2}, P') \\
&\times S(q+\frac{1}{2}(P'-P))\Gamma^\mu(q+\frac{1}{2}(P'-P), q-\frac{1}{2}(P'-P))S(q-\frac{1}{2}(P'-P)) \\
&\times \Gamma(q-\frac{1}{2}P', P)S(q-\frac{1}{2}(P+P'))\Big\}
\end{aligned}
$$

# Electromagnetic form factor

Quark–quark–photon vertex: Ball–Chiu vertex [10, 11]

$$\Gamma^\mu(p', p) =$$
$$\frac{1}{2}[A(-p'^2) + A(-p^2)]\gamma^\mu + \frac{(p'+p)^\mu}{(p'^2 - p^2)}\Big\{[A(-p'^2) - A(-p^2)]\frac{(\not{p}' + \not{p})}{2}$$
$$- [B(-p'^2) - B(-p^2)]\Big\}$$

Ward–Takahasi identity:

$$(p' - p)_\mu \Gamma^\mu(p', p) = S^{-1}(p') - S^{-1}(p)$$

For *Ansatz* of Mello, Melo, and Frederico [2]

$$\Gamma^\mu(p', p) = \gamma^\mu - \frac{m^3(p'^\mu + p^\mu)}{(p'^2 - \lambda^2)(p^2 - \lambda^2)}$$

# Electromagnetic form factor

Mello: $B(x) = M(x)|_{m0=0} = -m^3 \left[x - \lambda^2 + i\varepsilon\right]^{-1}$

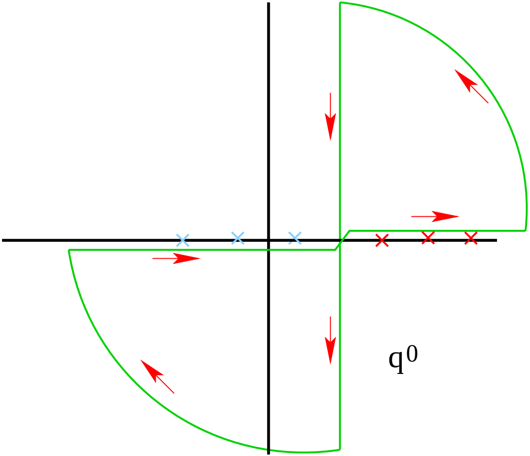
(1) Calculation using FeynCalc [5, 6] and Package-X [7, 8]
(2) “Euclidean” integration

- 3 nontrivial integrations (over $q^0$, $\xi$, $\vartheta$)

(3) Minkowski space integration utilizing light-cone momenta

# Electromagnetic form factor

Wick rotation: $q^0 = (q^0)_c - iq_4, \quad -\infty < q_4 < +\infty$



# Electromagnetic form factor

Poles of the integrand:

$$(q_0)_{1,2} = \mp\sqrt{M_q^2 + \xi^2 - \xi\sqrt{Q^2}\cos\vartheta + Q^2/4}$$

$$(q_0)_{3,4} = \mp\sqrt{M_q^2 + \xi^2 + \xi\sqrt{Q^2}\cos\vartheta + Q^2/4}$$

$$(q_0)_{5,6} = \frac{1}{2}\left(\sqrt{4M_\pi^2 + Q^2} \mp 2\sqrt{M_q^2 + \xi^2}\right)$$

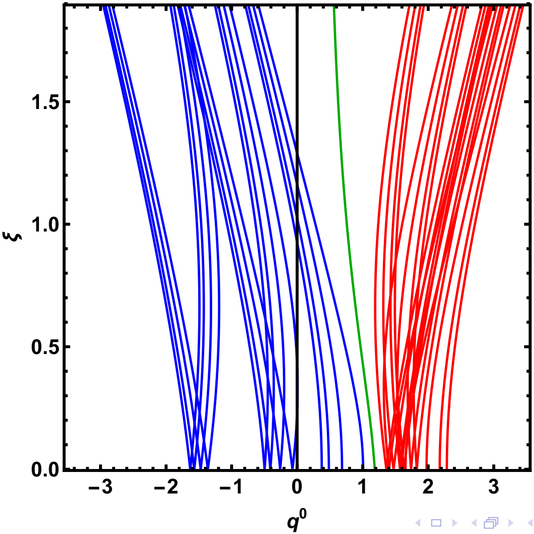
$$(q_0)_{7,8} = \frac{1}{4}\left(\sqrt{4M_\pi^2 + Q^2} \mp \sqrt{16M_q^2 + 16\xi^2 + 8\xi\sqrt{Q^2}\cos\vartheta + Q^2}\right)$$

$$(q_0)_{9,10} = \frac{1}{4}\left(\sqrt{4M_\pi^2 + Q^2} \mp \sqrt{16M_q^2 + 16\xi^2 - 8\xi\sqrt{Q^2}\cos\vartheta + Q^2}\right)$$

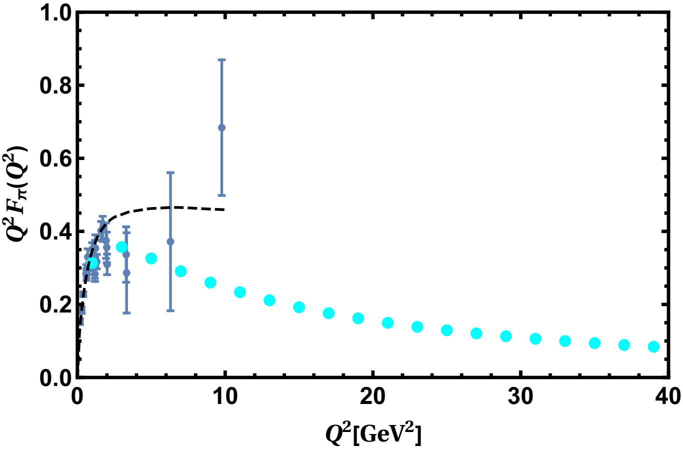
$$M_q^2 \in \{a_1, a_2, a_3, \lambda^2\}$$

# Electromagnetic form factor

$\vartheta = \frac{\pi}{3}$

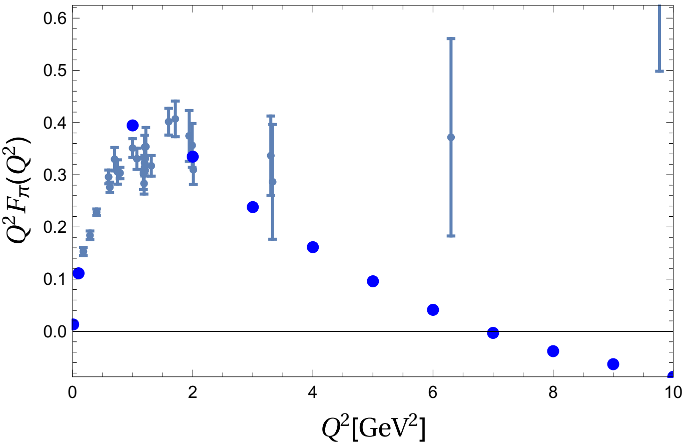


# Electromagnetic form factor



# Electromagnetic form factor

Calculation using 3R Quark Propagator [4]:



# Electromagnetic form factor

Asymptotic behavior, perturbative QCD [12, 13, 14]:

$$Q^2\, F_\pi(Q^2) = 16\pi\alpha_s(Q^2)\, f_\pi^2 \quad \text{for} \quad Q^2 \to \infty$$

where

$$\alpha_s(Q^2) \propto \frac{1}{\ln(Q^2/\Lambda^2_{\mathrm{QCD}})}\,.$$

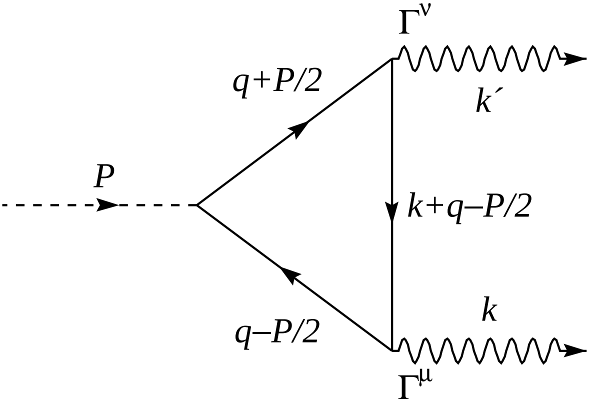
Asymptotic form of the dressed quark mass function:

$$M(p^2) \propto \begin{cases} \frac{1}{Q^2}[\ln(Q^2/\Lambda^2_{\mathrm{QCD}})]^{d-1} & m = 0 \\ \ln(Q^2/\Lambda^2_{\mathrm{QCD}})^{-d} & m \neq 0 \end{cases}$$

where $d = \dfrac{12}{33 - 2N_f}$.

Mello: $M(Q^2) \sim m_0$ 3R: $M(Q^2) \sim c/Q^2$

# Transition form factor




$+ (k \leftrightarrow k', \mu \leftrightarrow \nu)$

# Transition form factor

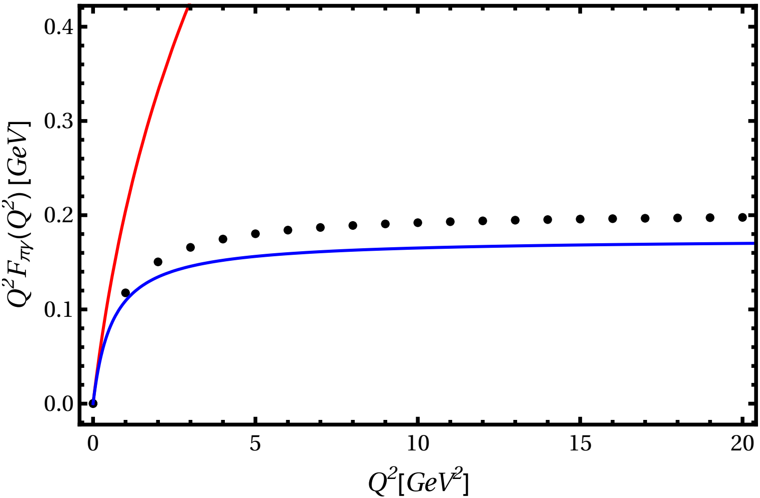
$$
\begin{aligned}
&T^{\mu\nu}(k,k') \\
&= i\int d^4x\, e^{ik\cdot x}\langle 0|T\{J^\mu(x),J^\nu(0)\}|\pi^0(P)\rangle \\
&= -N_c\,\frac{Q_u^2-Q_d^2}{2}\int\frac{d^4q}{(2\pi)^4}\mathrm{tr}\{\Gamma^\mu(q-\frac{P}{2},k+q-\frac{P}{2})S(k+q-\frac{P}{2}) \\
&\times\Gamma^\nu(k+q-\frac{P}{2},q+\frac{P}{2})S(q+\frac{P}{2})\left(-\frac{2B(-q^2)}{f_\pi}\gamma_5\right)S(q-\frac{P}{2})\} \\
&+(k\leftrightarrow k',\mu\leftrightarrow\nu) \\
&= \varepsilon^{\alpha\beta\mu\nu}k_\alpha k'_\beta T(k^2,k'^2)
\end{aligned}
$$

Transition form factor:

$$F_{\pi\gamma}(Q^2) = |T(-Q^2,0)|$$

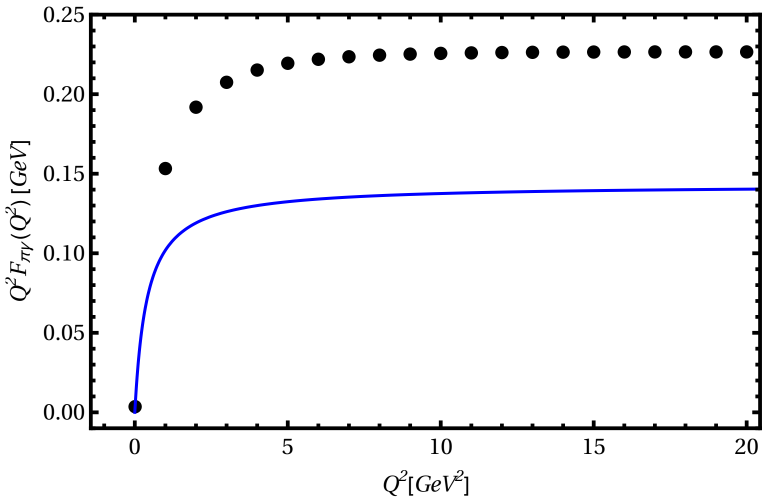
# Transition form factor

Mello: $B(x) = M(x)|_{m0=0} = -m^3 \left[x - \lambda^2 + i\varepsilon\right]^{-1}$



# Transition form factor

## 3R Quark Propagator [4]:



# Transition form factor

- Blue line: Brodsky–Lepage interpolation formula [15, 16]:

$$F_{\pi\gamma}(Q^2) = \frac{1}{4\pi^2 f_\pi}\left(1 + \frac{Q^2}{8\pi^2 f_\pi^2}\right)^{-1}$$

Two limiting values:

$$lim_{Q^2\to 0} F_{\pi\gamma}(Q^2) = \frac{1}{4\pi^2 f_\pi}$$
$$lim_{Q^2\to\infty} Q^2\, F_{\pi\gamma}(Q^2) = 2 f_\pi$$

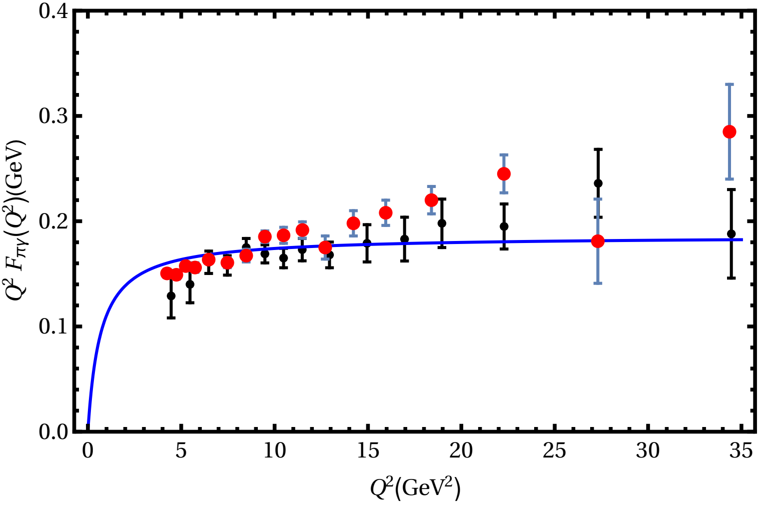
- Red line: Simple constituent quark model: $S^{-1}(q) = \not{q} - M_q$

$$T(k^2, k'^2) = \frac{M_q^2}{2\pi^2 f_\pi} C_0(k^2, k'^2, M_\pi^2, M_q^2, M_q^2, M_q^2)$$

$$T(0, -Q^2) \sim \frac{M_q^2}{2\pi^2 f_\pi}\, \frac{1}{2Q^2} \ln^2\left(\frac{M_q^2}{Q^2}\right) \quad \text{for} \quad Q^2 \to \infty$$

# Transition form factor

Experimental data: • Belle [17] and • BaBar [18] collaborations



# Conclusions

- These two models have very simple analytic structure:

$$S(q) = -\boxed{\sigma_V}(-q^2)\not{q} - \boxed{\sigma_S}(-q^2)$$

$$S^{-1}(q) = \boxed{A}(-q^2)\not{q} - \boxed{B}(-q^2)$$

  Quark dressing functions have only simple poles for $q^2 \in \mathbb{R}$, $q^2 > 0$.

- "Minkowski" and "Euclidean" integrations are equivalent
- The *Ansätze* don't behave properly in the UV limit
  $\Rightarrow$ one can hardly expect that $F_\pi(Q^2)$ or $F_{\pi\gamma}(Q^2)$ will have correct high–$Q^2$ limit.
- For low and intermediate energies these models could suffice.
- For proper UV behavior some "Log" dependence is necessary
  $\Rightarrow$ branch cuts appearance $\Rightarrow$ much more complications

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