Group theory and SM extensions

based on: **arXiv1909.02208**, *APP***B52**(2021)63, *J.Phys.***A50**(2017)115401

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Intention of the talk

- The talk will not be about standard Lie group / Lie algebra theory on GUT.
  That is based on standard Dynkin classification of **simple Lie algebras**.
  (That is classic, since $\sim$1950's, one can learn that on standard lectures.)

  Note:
  Idea of Dynkin classif.: how one can inject $\mathfrak{SU}(2)$ generators into simple Lie algebras.
  $\longrightarrow$ Dynkin diagrams etc.
  Overview literature on standard GUT:
  R.Slansky: Group Theory for Unified Model Building; *Phys.Rept.*79(1981)1-128.
  E.Witten: Quest for Unification; **`arXiv:hep-ph/0207124`**

- This talk will take a more general point of view.
  Not all Lie groups / Lie algebras are simple or semisimple!
  The talk will revolve around:
  Levi decomposition theorem of any general finite dim Lie algebras.
  With this, one can understand SUSY and more ...

- The material of the talk is covered by: **`arXiv1909.02208`**

Motivation

- **Symmetry unification could simplify SM+GR.** At present: zillions of couplings. Unified symmetry group can reduce independent couplings.

- **Usual GUTs come at cost.** E.g. embedding $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ into $\mathfrak{su}(5)$:

$$\begin{pmatrix} \mathfrak{su}(2) & * \\ * & \mathfrak{su}(3) \end{pmatrix} \quad \longleftarrow \text{(lot of off diagonal gauge fields)}$$

(in Georgi–Glashow)
What about the off-diagonals? Lot of exotic gauge fields $\longrightarrow$ proton decay etc.

Question:

- **Can't one eliminate off-diagonals in some models?** Not easy, but with some trick, one can. (See this talk.)

- **Age-old idea: can one also involve spacetime?** Could one also incorporate internal + spacetime symmetry group in a unified group? (Extremist point of view.)

- **Internal–vs–spacetime symm unification no-go theorems.** Spacetime symmetries (Poincaré group) and compact internal symmetries (compact gauge group) cannot be unified in a straightforward way (McGlinn1964, Coleman–Mandula1967).

- **Supersymmetry (SUSY).** The no-go theorems are circumventable if some amount of "exotic" symmetries are allowed (Haag–Lopuszanski–Sohnius1975).

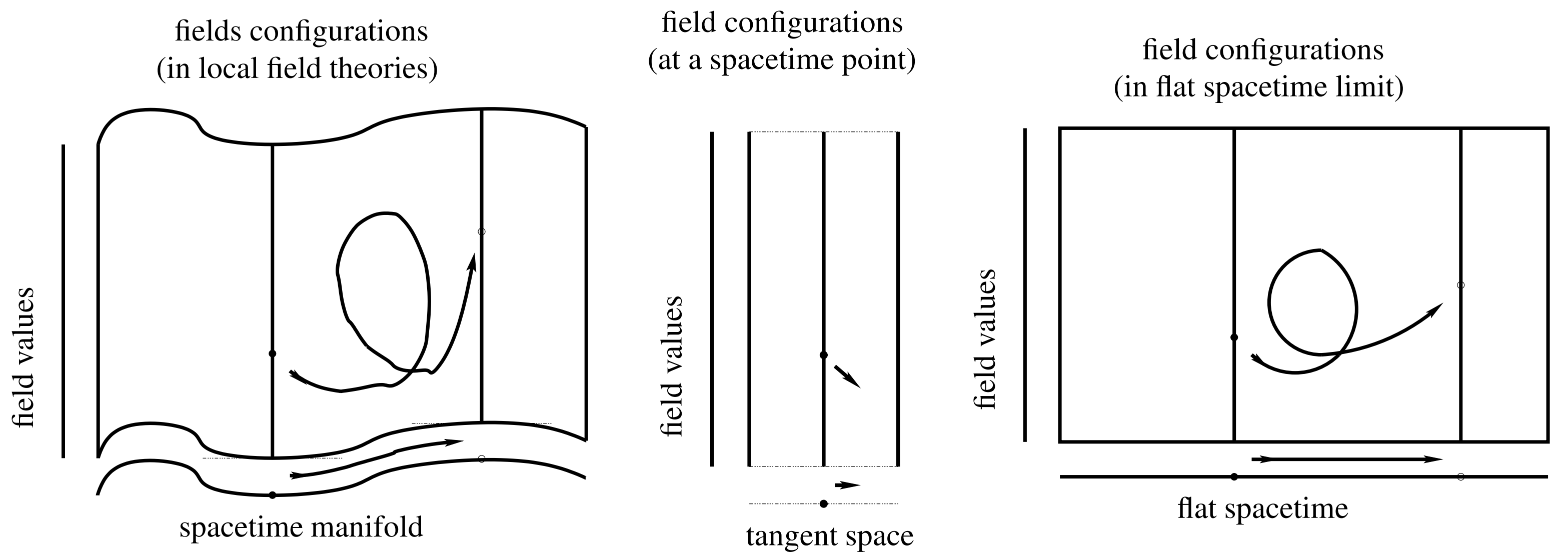
- **SUSY is not seen experimentally.** At present status (2021).

Question:

- **Do math. alternatives to SUSY exist for gauge+spacetime symmetry unification?** (Yes, and even without the problem of off-diagonals, see this talk.)
- **Most general constraints?** What are the most general (strongest) Lie algebra theoretical obstacles to unifications, and how can one bypass them?

Assume that we are looking for a model which has a classical field theory limit.

In a (classical) field theory we have finite degrees of freedom at points of 4d spacetime.



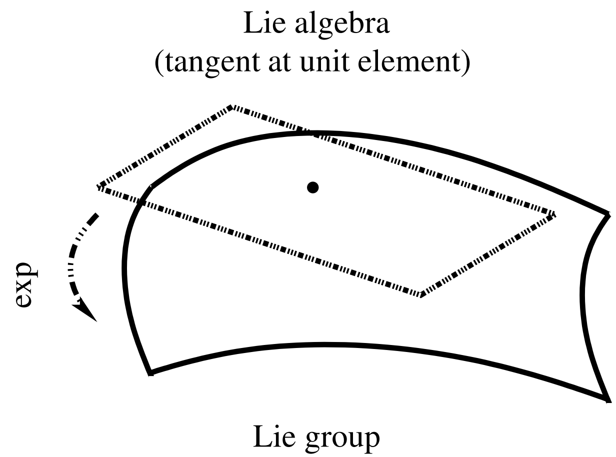
Diffeomorphism inv. Lagrangian is constrained by *first order symm.* at points of spacetime.

Such symmetry generators are finitely many, because they act on finite degrees of freedom.
They have the same structure as if we looked at global symmetries limit.
($\sim$ symmetries at special relativistic limit.)

Such symmetries form always finite dimensional real Lie algs/groups. (Not a big surprise.)
So, looking at general structure of Lie groups / Lie algs seems a wise strategy.

General structure of finite dim real Lie groups / algs

- **Group.** A collection of transformations, which can be composed, inverted, and there is unit transformation within the collection. (All groups arise in this way.)
- **Lie group.** A parametric group, parametrized by a finite collection of real parameters. E.g.: rotation group, symmetry group of flat plane, Poincaré group, $\mathrm{SU}(N)$ etc.
- **Lie algebra.** Derivatives (or, equivalently, the tangent) of a Lie group at the unit element.



Thus, the Lie algebra is the infinitesimal version of the Lie group.
Exponential map makes a Lie group element from Lie algebra element (generator).
Lie algebra completely characterizes Lie group, modulo global topology.

- **Ado's theorem.** Every Lie algebra can be realized as matrix algebra.
(This is not always true for Lie groups!)

- **Universal covering group.** Every Lie group has a corresponding connected, simply connected Lie group which is isomorphic to it around the unity. $\longleftrightarrow$ Lie algebra.

- **We only consider Lie algebra level in this talk.** This tells already a lot about possible group embeddings and isomorphisms.
(Necessary condition. Sufficient for connected and simply connected groups.)

- **General notation for a finite dim real Lie algebra:** $\mathfrak{e}$.
- **Lie subalgebra.** A linear subspace over which Lie $[\,,\,]$ closes.
- **Normal Lie subalgebra, or ideal.** A subspace $\mathfrak{i} \subset \mathfrak{e}$ is ideal, whenever $[\mathfrak{e}, \mathfrak{i}] \subset \mathfrak{i}$.
  E.g. the translation generators for the symmetry group of flat plane.
- **Extension, or semi-semi-direct sum.** Just synonym to above. If $\mathfrak{i}$ is ideal, and $\mathfrak{c}$ is complementing linear subspace to $\mathfrak{i}$, then we can write $\mathfrak{e} = \mathfrak{i} + \mathfrak{c}$.
- **Semi-direct sum.** As above, if $\mathfrak{c}$ closes as Lie subalgebra. Notation: $\mathfrak{e} = \mathfrak{i} \oplus\!\!\!\!+ \mathfrak{c}$.
  E.g. the symm generators of flat plane is semi-direct sum of translations and rotations.
- **Direct sum.** As above, but $\mathfrak{c}$ is also ideal. Notation: $\mathfrak{e} = \mathfrak{i} \oplus \mathfrak{c}$ or $\mathfrak{c} \oplus \mathfrak{i}$.
  Direct sum means that the large Lie alg is built of completely independent parts $\mathfrak{i}$, $\mathfrak{c}$.
  E.g. Standard Model gauge group Lie alg $\mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3)$.
  GUT strategy tries to avoid direct sum (requirement of direct-indecomposability).

Killing form:

On any finite dim real Lie algebra one has the *Killing form*, an invariant scalar product.

$$x \cdot y \quad = \quad \mathrm{Tr}\,(\mathrm{ad}_x\,\mathrm{ad}_y) \qquad\qquad (\text{ with notation } \mathrm{ad}_x\, z := [x, z]\ )$$

(It appears e.g. in the Yang–Mills Lagrangians:

$$F_{\mu\nu} \cdot F^{\mu\nu}$$

$\uparrow$ Killing form)

It may be definite, indefinite, or even can be degenerate:

$$\begin{pmatrix} (+1) & & \\ & (-1) & \\ & & (0) \end{pmatrix}$$

} (nondegenerate directions)

$\longleftarrow$ (degenerate directions)

- **Levi–Mal'cev decomposition theorem**:

$$\underbrace{\mathfrak{e}}_{\text{finite dim real Lie alg}} = \underbrace{\mathfrak{r}}_{\substack{\text{Killing form degenerates} \\ (\textit{radical}, \text{ or } \textit{solvable part})}} \oplus \underbrace{\mathfrak{l}}_{\substack{\text{max. Lie subalg. with nondegenerate Killing form} \\ (\textit{Levi factor}, \text{ or } \textit{semisimple part})}}$$

  E.g.: the symmetries of flat plane (translations $\oplus$ rotations) is a typical example.

- **Lie alg of Poincaré group** (example):

$$\underbrace{\mathfrak{p}}_{\text{Poincaré alg}} = \underbrace{\mathfrak{t}}_{\text{translations (radical)}} \oplus \underbrace{\ell}_{\text{homogeneous Lorentz generators (Levi factor)}}$$

  is a typical demonstration of Levi's decomposition theorem.

- **Traditional gauge theory folklore:** only (semi)simple Lie algs important. $\mathfrak{su}(N)$ etc.

How complicated a radical $\mathfrak{r}$ can be?

- **most generally: $\mathfrak{r}$ is solvable** $\Leftrightarrow$ for the Lie algebra $\mathfrak{r}$, the sequence

  $\mathfrak{r}^{(0)} := \mathfrak{r}, \quad \mathfrak{r}^{(1)} := [\mathfrak{r}^{(0)}, \mathfrak{r}^{(0)}], \quad \ldots, \quad \mathfrak{r}^{(k+1)} := [\mathfrak{r}^{(k)}, \mathfrak{r}^{(k)}] = \{0\}$ for finite k.

- **special case: $\mathfrak{r}$ is nilpotent.** There is finite k such that

  $\mathrm{ad}_{x_1} \ldots \mathrm{ad}_{x_k} = 0$ holds for all $x_1, \ldots, x_k \in \mathfrak{r}$. ($\Leftrightarrow$ ad_x is nilpotent for all $x \in \mathfrak{r}$.)

- **even more special: $\mathfrak{r}$ is abelian.** One has that

  $\mathrm{ad}_x = 0$ holds for all $x \in \mathfrak{r}$.

(Solvable or even nilpotent Lie algebras are known to be too many to classify!)

How complicated Levi factor $\mathfrak{l}$ can be?

$$\underbrace{\mathfrak{e}}_{\text{finite dim real Lie alg}} = \underbrace{\mathfrak{r}}_{\substack{\text{Killing form degenerates} \\ (\textit{radical}, \text{ or } \textit{solvable part})}} \subsetplus \underbrace{\overbrace{\mathfrak{l}_1}^{(\textit{simple})} \oplus \ldots \oplus \overbrace{\mathfrak{l}_n}^{(\textit{simple})}}_{\substack{\text{max. Lie subalg. with nondegenerate Killing form} \\ (\textit{Levi factor}, \text{ or } \textit{semisimple part})}}$$

- The Levi factor (semisimple part) is direct product of **simple** parts: $\mathfrak{l}_i$ $(i = 1, \ldots, n)$ are themselves semisimple and have no ideals within.

- Simple Lie algebras a completely classified (complete list available): $\mathfrak{su}(N)$, $\mathfrak{sl}(2, \mathbb{C})$, etc (via Dynkin diagrams: A_n, B_n, C_n, D_n, $E_{6,7,8}$, F_4, G_2).

- **Lie alg of Poincaré group** (example):

$$\underbrace{\mathfrak{p}}_{\text{Poincaré Lie alg}} = \underbrace{\mathfrak{t}}_{\text{translation generators (radical)}} \subsetplus \underbrace{\overbrace{\boldsymbol{\ell}}^{(\textit{simple})}}_{\text{Lorentz generators (Levi factor)}}$$

$\boldsymbol{\ell} \cong \mathfrak{sl}(2, \mathbb{C})$

Lie algebra of super-Poincaré group (SUSY):

$$\underbrace{\mathfrak{p}_s}_{\text{super-Poincaré gens}} = \underbrace{\mathfrak{s}}_{\text{supertranslation gens (radical)}} \oplus \underbrace{\boldsymbol{\ell}}_{\text{Lorentz gens (Levi factor)}}$$

is a similar example, with a bit larger radical (so called: two-step nilpotent).

Supertranslations: a transformation group on the vector bundle of superfields. Action:

$$\begin{pmatrix} \theta^A \\ x^a \end{pmatrix} \xmapsto{\begin{pmatrix} \epsilon^A \\ d^a \end{pmatrix}} \begin{pmatrix} \theta^A + \epsilon^A \\ x^a + d^a + \sigma^a_{AA'}\,\mathrm{i}\big(\theta^A \bar{\epsilon}^{A'} - \epsilon^A \bar{\theta}^{A'}\big) \end{pmatrix}$$

on the "supercoordinates" and the affine spacetime coordinates.

Often, SUSY is presented as “super-Lie algebra”:

$$
\begin{array}{lr}
\checkmark & [\, P_a \;, P_b \,] = 0, \\
\checkmark & [\, P_a \;, Q_A \,] = 0, \\
\checkmark & [\, P_a \;, \bar{Q}_{A'} \,] = 0, \\
\textbf{!!!} \rightarrow & \{\, Q_A \;, Q_B \,\} = 0, \\
\textbf{!!!} \rightarrow & \{\, \bar{Q}_{A'} \;, \bar{Q}_{B'} \,\} = 0, \\
\textbf{!!!} \rightarrow & \{\, Q_A \;, \bar{Q}_{A'} \,\} = 2\,\sigma^a_{AA'}\, P_a.
\end{array}
$$

If not a Lie algebra, how can it be a collection of infinitesimal transformations? Answer:

[Nucl.Phys.**B76**(1974)477, Phys.Lett.**B51**(1974)239]:
Take $\epsilon^A_{(i)}$ $(i = 1, 2)$ “supercoordinate” (Grassmann valued two-spinor) basis.
Introduce new generators $\delta_{(i)} = \epsilon^A_{(i)}\, Q_A$ instead of Q_A. (Infinitesimal change of superfields.)
$\Rightarrow$ SUSY has also an ordinary finite dim real Lie algebra presentation.

Exponentiating this Lie algebra: super-Poincaré Lie group is obtained. SUSY is not so exotic!
Ordinary Lie group / Lie algebra theory also applies!

SUSY viewed as an ordinary Lie group

The super-Poincaré can be viewed as an ordinary Lie group, with Lie algebra:

(nonvanishing adjoint action)

$$\underbrace{\mathfrak{p}_s}_{\text{super-Poincaré}} = \underbrace{\underbrace{\left(\underbrace{\mathfrak{t}}_{\text{ideal of translations}} + \underbrace{q}_{\text{pure supertranslations}} \right)}_{=\mathfrak{S}\ \text{(ideal of all supertranslations)}} \subset\!\!\!\!+ \underbrace{\boldsymbol{\ell}}_{\text{Lorentz generators}}}_{\text{super-Poinceré}}$$

Indecomposable (unified).

With super-Lie algebra, one can encode certain ordinary Lie algebras (Lie groups) which have the structure

$$\underbrace{\left(\mathfrak{t} \;+\; q\right)}_{=\mathfrak{s}} \;\oplus\; \mathfrak{l}$$

with

- $\mathfrak{l}$ being a Lie subalgebra,
- $\mathfrak{s}$ being a complementing ideal,
- $\mathfrak{t}$ being a ideal within $\mathfrak{s}$, and
- $\mathfrak{t}$ commutes with all $\mathfrak{s}$ and $q \cong \mathfrak{s}/\mathfrak{t}$ is abelian. $\leftarrow$!!!
  (translations commute with all supertranslations,
  supertranslations without considering contribution of translations are abelian.)

$\uparrow$

This makes it possible to switch the sign of “odd” part (q) with Grassmannian basis.

All possible extensions of the Poincaré group

O'Raifeartaigh theorem (1965) — all possible finite dim extensions of the Poincaré Lie alg:

- **Either:**

$$\mathfrak{e} = \mathfrak{r} \oplus \mathfrak{l}_1 \oplus \ldots \oplus \mathfrak{l}_n$$

$$\mathfrak{p} = \mathfrak{t} \oplus \ell$$

(A) Trivial extension: ∼Coleman–Mandula.

(*Other symmetries are independent from Poincaré.*)

(B) Extended radical: not (A), radical bigger than $\mathcal{T}$.

(SUSY, extended SUSY, and our new example.)

- **Or:**

$$\mathfrak{e} = \mathfrak{r} \oplus \mathfrak{l}_1 \oplus \ldots \oplus \mathfrak{l}_n$$

$$\mathfrak{p} = \mathfrak{t} \oplus \ell$$

(C) Poincaré embedded into simple Lie group.

(*Conform* — $\mathfrak{so}(2,4)$ — *theories etc.*

Heavy symmetry breaking needed for a

gauge-theory-like limit, i.e. to point out spacetime.)

Group theoretical constraints for any unification

- Collection of all symmetries:

$$\underbrace{\mathfrak{e}}_{\text{Lie alg}} = \underbrace{\mathfrak{r}}_{\substack{\text{Killing form degenerates} \\ (\textit{radical}, \text{ or } \textit{solvable part})}} \oplus \underbrace{\mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_n}_{\substack{\text{max. Lie subalg. with nondeg. Killing form} \\ (\textit{Levi factor}, \text{ or } \textit{semisimple part})}}$$

- Poincaré group:

$$\underbrace{\mathfrak{p}}_{\text{Poincaré Lie alg}} = \underbrace{\mathfrak{t}}_{\text{translations (radical)}} \oplus \underbrace{\ell}_{\text{homogeneous Lorentz gens (Levi factor)}}$$

- Compact gauge group:

$$\underbrace{\mathfrak{g}}_{\text{compact Lie alg}} = \underbrace{\mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1)}_{\text{compact abelian part (radical)}} \oplus \underbrace{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m}_{\text{compact non-abelian part (Levi factor)}}$$

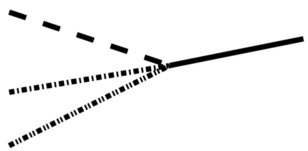
How (extended) SUSY works Lie group theoretically?

- **Unification via *extended super-Poincaré group*:**

(nonvanishing adjoint Lie subalg action)

$$\underbrace{\underbrace{\Big(\big(\underbrace{\mathfrak{t}}_{\text{translations}} \oplus \underbrace{\mathfrak{z}}_{\text{central charges}}\big) + \underbrace{q_{\mathrm{ext}}}_{\text{pure supertranslations}}\Big)}_{\text{all supertranslations (radical)}} \niplus \Big(\underbrace{\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m}_{\text{compact internal symmetries}} \oplus \underbrace{\ell}_{\text{Lorentz}}\Big)}_{\text{Lie alg of extended super-Poincaré group}}$$

- **The extended super-Poincaré group is direct-indecomposable (unified).**
  $\Rightarrow$ Connects spacetime symmetries with compact internal (gauge) symmetries.
  $\Rightarrow$ Connects potentially independent compact internal symmetries with each-other.
  $\Rightarrow$ Running of coupling factors do unify.



Running of gauge couplings

- **Operated by O'Raifeartaigh theorem case B.** Via the extension of the radical.

Recent knowledge on generic Lie algebra structure

L.Šnobl: *J.Phys.***A43**(2010)505202 gives a structural theorem.
L.Šnobl,P.Winternitz: *Classification and identification of Lie algebras* (AMS, 2014).

(nonvanishing, faithful, adjoint action)

$$\underbrace{\mathfrak{e}}_{} \; = \; \underbrace{m_1 + \ldots + m_k}_{} \; + a \; \oplus \; \underbrace{\underbrace{\mathfrak{l}_1}_{} \oplus \ldots \oplus \underbrace{\mathfrak{l}_n}_{}}_{}$$

$\mathfrak{e}$: finite dim real indecomposable Lie alg

$\mathfrak{n}$, max. normal Lie subalg. with nilpotent gens (Killing form=0) (*nilradical*)

$\mathfrak{l}_1$, ..., $\mathfrak{l}_n$: (*simple*)

$\mathfrak{r}$, Killing form degenerates (*radical*, or *solvable part*)

$\mathfrak{l}$, max. Lie subalg. with nondegenerate Killing form (*Levi factor*, or *semisimple part*)

with

$$\mathfrak{r} = \mathfrak{n} + a \quad , \qquad \mathfrak{n} = m_1 + \ldots + m_k \quad , \qquad m_{j-1} \subset [m_k, m_j] \quad , \qquad \mathrm{ad}_{\mathfrak{l}}\, m_j \subset m_j \quad ,$$

and $\mathfrak{l}$ acts on m_k faithfully.

(m_j are the "levels" of the nilpotent Lie alg $\mathfrak{n}$, with $j = 1, \ldots, k$.)

If one factors out nilradical $\mathfrak{n} = m_1 + \ldots + m_k$ from $\mathfrak{e}$:

(nonvanishing, faithful, adjoint action)

$$\underbrace{\mathfrak{e}/\mathfrak{n}}_{\text{when factored with nilradical}} = \underbrace{\underbrace{\cancel{m_1 + \ldots + m_k}}_{\mathfrak{n}\text{, max. normal Lie subalg. with nilpotent gens (Killing form=0) (nilradical)}} + a}_{\mathfrak{r}\text{, Killing form degenerates (radical, or solvable part)}} \oplus \underbrace{\underbrace{\mathfrak{l}_1}_{\text{(simple)}} \oplus \ldots \oplus \underbrace{\mathfrak{l}_n}_{\text{(simple)}}}_{\mathfrak{l}\text{, max. Lie subalg. with nondegenerate Killing form (Levi factor, or semisimple part)}}$$

$\Downarrow$

$$\underbrace{\mathfrak{e}/\mathfrak{n}}_{\text{when factored with nilradical}} = \underbrace{a}_{\text{abelian (radical)}} \oplus \underbrace{\underbrace{\mathfrak{l}_1}_{\text{(simple)}} \oplus \ldots \oplus \underbrace{\mathfrak{l}_n}_{\text{(simple)}}}_{\mathfrak{l}\text{, max. Lie subalg. with nondegenerate Killing form (Levi factor, or semisimple part)}}$$

So, if we forbid nilradical (no nilpotent charges):

$$\underbrace{\mathfrak{e}/\mathfrak{n}}_{\substack{\text{when factored}\\ \text{with nilradical}}} = \underbrace{a}_{\substack{\text{abelian}\\ (\textit{radical})}} \oplus \underbrace{\underbrace{\mathfrak{l}_1}_{(\textit{simple})} \oplus \ldots \oplus \underbrace{\mathfrak{l}_n}_{(\textit{simple})}}_{\mathfrak{l}}$$

$a \cong \mathfrak{u}(1) \oplus \ldots \oplus \mathfrak{u}(1)$

$\mathfrak{l}$, max. Lie subalg. with nondegenerate Killing form (*Levi factor*, or *semisimple part*)

E.g. Standard Model structure group: $\mathfrak{u}(1) \quad \oplus \quad \mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{sl}(2, \mathbb{C})$

Such Lie algebras are called **reductive**. Equivalent defining properties:

- its adjoint rep is completely reducible (direct sum of irred ones),
- has any faithful finite dim completely reducible reps,
- its radical commutes with everybody,
- it is direct sum of abelian and semisimple Lie algs.

The only invariant scalar products: trace forms (as in ordinary gauge theory, $F_{\mu\nu} \cdot F^{\mu\nu}$).

Lie algs of compact Lie groups are always reductive.

(In general: reducibility $\not\Rightarrow$ complete reducibility. Automatic only for reductives.)

So, for generic (non-reductive), we had:

(nonvanishing, faithful, adjoint action)

$$\underbrace{\mathfrak{e}}_{\text{finite dim real indecomposable Lie alg}} = \underbrace{\underbrace{m_1 + \ldots + m_k}_{\mathfrak{n}} + a}_{\mathfrak{r}} \oplus \underbrace{\underbrace{\mathfrak{l}_1}_{(simple)} \oplus \ldots \oplus \underbrace{\mathfrak{l}_n}_{(simple)}}_{\mathfrak{l}}$$

$\mathfrak{n}$, max. normal Lie subalg. with nilpotent gens (Killing form=0) (*nilradical*)

$\mathfrak{r}$, Killing form degenerates (*radical*, or *solvable part*)

$\mathfrak{l}$, max. Lie subalg. with nondegenerate Killing form (*Levi factor*, or *semisimple part*)

$\cong \mathfrak{u}(1) \oplus \ldots \oplus \mathfrak{u}(1)$

Generally: $\mathfrak{r} = \mathfrak{n} + a$.

Under mild assumptions (Mostow, Šnobl) a closes as Lie subalgebra. Then: $\mathfrak{r} = \mathfrak{n} \oplus \mathfrak{a}$.

(nonvanishing, faithful, adjoint action)

$$\mathfrak{e} = \underbrace{m_1 + \ldots + m_k}_{\mathfrak{n}} \oplus \underbrace{\underbrace{\mathfrak{a}}_{(abelian)} \oplus \underbrace{\mathfrak{l}_1}_{(simple)} \oplus \ldots \oplus \underbrace{\mathfrak{l}_n}_{(simple)}}_{(reductive)}$$

$\mathfrak{n}$, max. normal Lie subalg. with nilpotent gens (Killing form=0) (*nilradical*)

(nonvanishing, faithful, adjoint action)

$$\mathfrak{e} = \underbrace{m_1 + \ldots + m_k}_{\substack{\mathfrak{n}\text{, max. normal Lie subalg.} \\ \text{with nilpotent gens (Killing form=0)} \\ (\textit{nilradical})}} \ltimes \underbrace{\underbrace{\mathfrak{a}}_{(\textit{abelian})} \oplus \underbrace{\mathfrak{l}_1}_{(\textit{simple})} \oplus \ldots \oplus \underbrace{\mathfrak{l}_n}_{(\textit{simple})}}_{(\textit{reductive})}$$

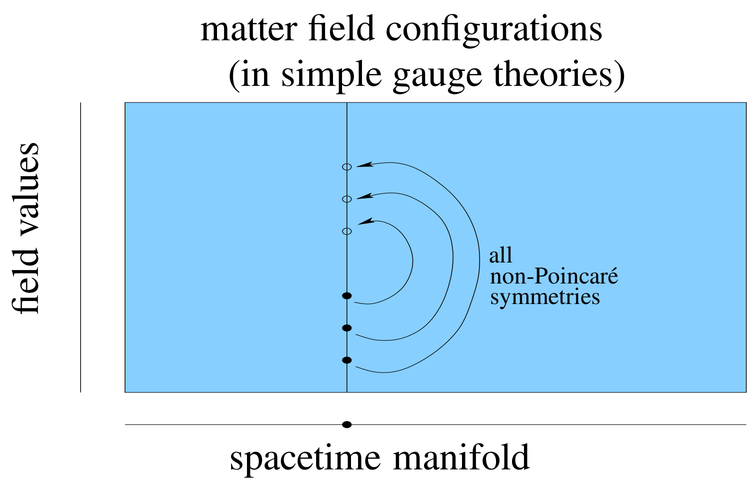
Lie alg of extended super-Poincaré group maxes it out (with two-step nilradical):

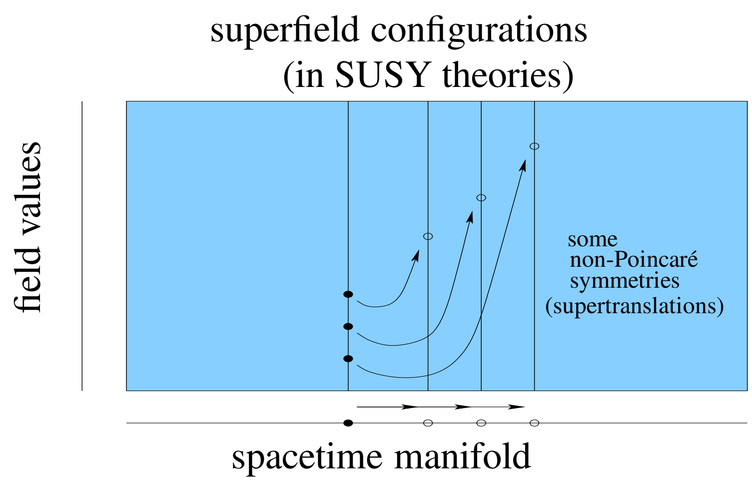
(nonvanishing adjoint Lie subalg action)

$$\underbrace{\underbrace{(\underbrace{\mathfrak{t}}_{\text{translations}} \oplus \underbrace{\mathfrak{z}}_{\text{central charges}})}_{m_1} + \underbrace{\underbrace{q_{\text{ext}}}_{\text{pure supertranslations}}}_{m_2}}_{\text{all supertranslations } (\textit{nilradical})} \ltimes \underbrace{\underbrace{\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m}_{\text{compact internal symmetries}} \oplus \underbrace{\ell}_{\text{Lorentz}}}_{(\textit{reductive})}$$

A curious property of (extended) super-Poincaré group

- **Not a vector bundle automorphism group.**

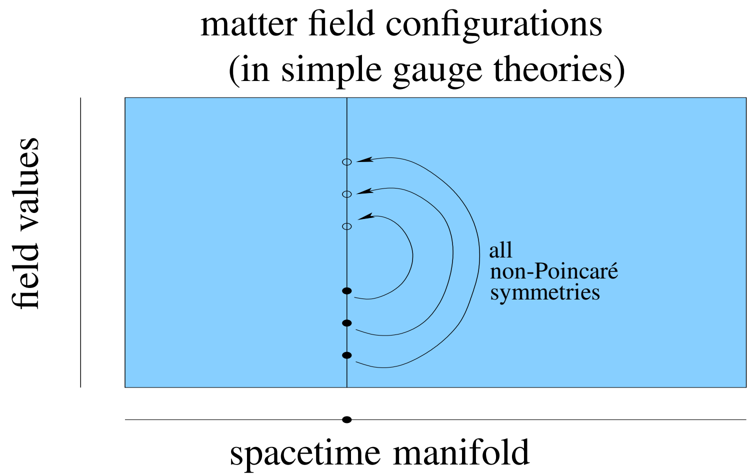


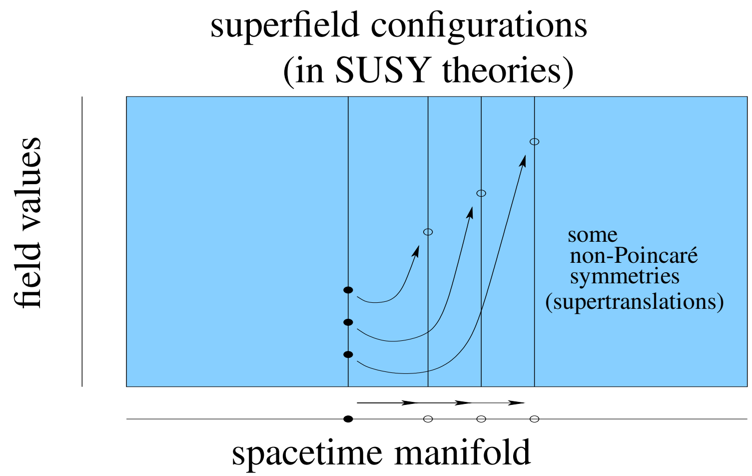


Supertranslations:

$$\begin{pmatrix} \theta^A \\ x^a \end{pmatrix} \xmapsto{\begin{pmatrix} \epsilon^A \\ d^a \end{pmatrix}} \begin{pmatrix} \theta^A + \epsilon^A \\ x^a + d^a + \sigma^a_{AA'} \mathrm{i}\big(\theta^A \bar{\epsilon}^{A'} - \epsilon^A \bar{\theta}^{A'}\big) \end{pmatrix}$$

- **Therefore, symmetry breaking needed.** Because in (extended) SUSY, the non-Poincaré symmetries couple too strongly to spacetime symmetries.



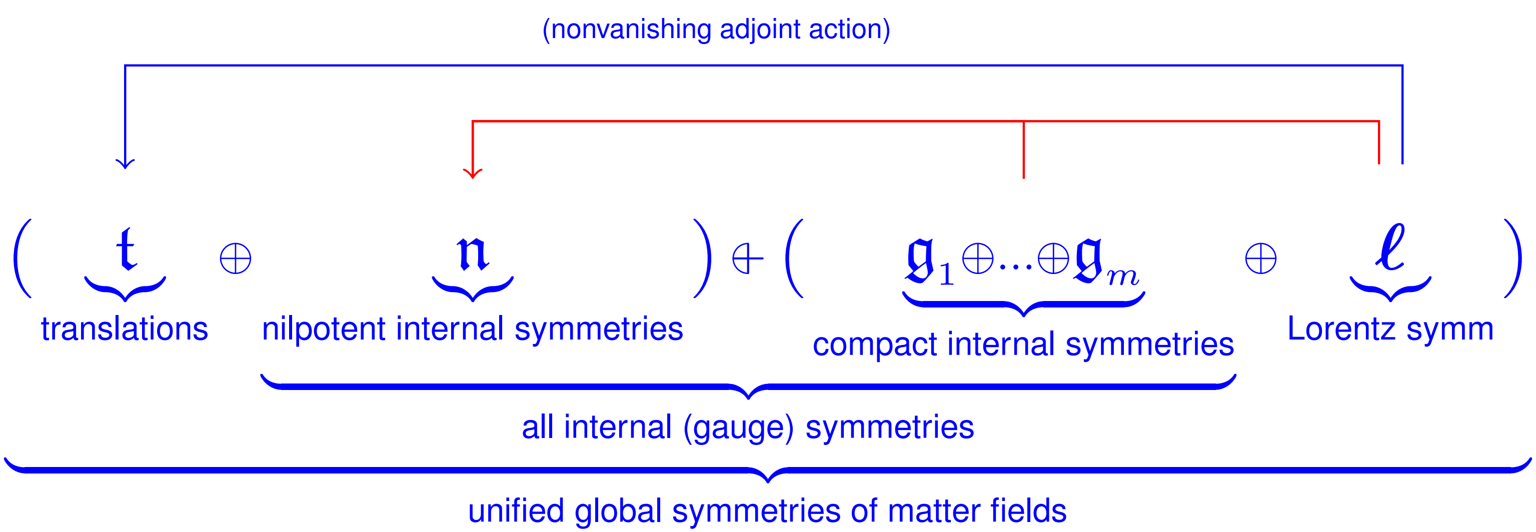


(Not a vector bundle automorphism group over spacetime.)

Experimental hint not seen for this, so symmetry breaking needed for gauge-theory-like limit. (bug? feature?)

A possible alternative mechanism to SUSY

- ***Conservative*** **extensions of the Poincaré group.**
  $\Leftrightarrow$ The non-Poincaré symmetries are really all *internal*, i.e. do not act on spacetime.
  $\Leftrightarrow$ There exists $\mathfrak{p} \xrightarrow{i} \mathfrak{e} \xrightarrow{o} \mathfrak{p}$ homomorphisms, such that $o \circ i =$ identity.
  $\Leftrightarrow$ Symm. breaking not needed for gauge-theory-like limit (vector bundle automorphism).
- **The (extended) super-Poincaré is non-conservative extension of Poincaré group.**
- **All possible conservative extensions of the Poincaré group:**

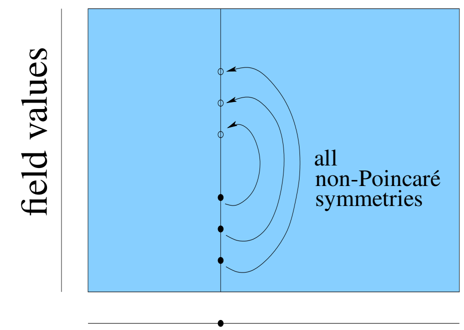


O'Raifeartaigh theorem + energy non-negativity $\Rightarrow$ these are only possible ones.
Similar gauge – spacetime symmetry unification to extended SUSY, via extended radical.

- **In a conservative Poincaré extension, non-Poincaré symmetries are all internal.**

any conservative extension of Poincaré Lie alg $=$

$\{$all internal symmetries$\}$ $\niplus$ $\{$Poincaré$\}$ $\Longleftrightarrow$



(SUSY does not admit this property.)

- **Constructed an example:**

  See **`arXiv1909.02208`** and *J.Phys.***A50**(2017)115401, with $\mathfrak{g}=\mathfrak{u}(1)$.
  It is the symmetry group acting on a QFT-inspired algebra valued fields.

  Price to pay: the full internal symmetry group is not purely $\{$compact$\}$ but

  $\{$nilpotent$\}$ $\niplus$ $\{$compact$\}$

  (Issue: nilpotent generators $\rightarrow$ corresp. gauge fields have zero YM kinetic Lagrangian!!)

In a *conservative* Poincaré group extension: there exists a homomorphism of

$$\underbrace{\underbrace{\mathfrak{n}}_{\text{nilpotent internal symmetries}} \subset\!\!\!\!+ \left(\underbrace{\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m}_{\text{compact internal symmetries}} \oplus \underbrace{\mathfrak{p}}_{\text{Poincaré symmetries}} \right)}_{\text{unified symm, acting on fundamental field degrees of freedom}} \longrightarrow \underbrace{\underbrace{\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m}_{\text{compact internal symmetries}} \oplus \underbrace{\mathfrak{p}}_{\text{Poincaré symmetries}}}_{\text{observed symmetries, acting on some derived field quantities which are function of fundamental degrees of freedom}}$$

No immediate contradiction with experimental situation.
(Nilpotent internal symmetries can act "hidden" in some fundamental d.o.f.)

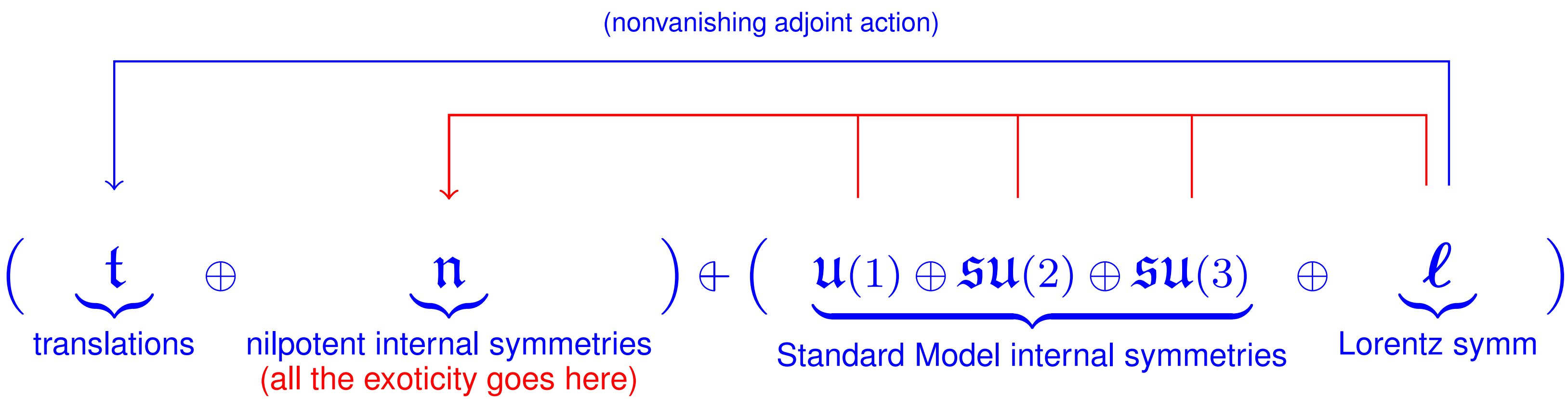
(Distant analogy:

$$\underbrace{\Psi}_{\mathfrak{u}(1) \oplus \text{Poincaré acts on it}} \longmapsto \underbrace{\text{Fierz bilinears of } \Psi}_{\text{only Poincaré acts on it}}$$

The Fierz bilinears forget the fundamental $\mathfrak{u}(1)$ symmetry of Dirac bispinor fields.
(But such "forgetting function" mechanism works also for semi-direct sum.))

Perspectives

Least exotic solution to gauge–spacetime and gauge–gauge symmetry unification: *conservative* unification pattern.



Unification happens not because of a heavy symmetry breaking.
But because of common adjoint subgroup action on "hidden" nilpotent internal symmetries.
Minimal exoticity: we inject subgroups where they naturally belong in Levi decomposition.
Unification achieved not by symmetry *breaking* but by symmetry *hiding*.

L.Šnobl, *J.Phys.***A43**(2010)505202: under mild conditions, max 1 copy of $\mathfrak{u}(1)$ present.

Eliminating nilpotent gauge fields

Presence of some nilpotent generators acting on the matter field sector could be plausible.

But what to do with corresponding gauge fields?

They have vanishing Yang–Mills kinetic term $\Rightarrow$ don't have kinetic energy, don't propagate.
($\mathrm{Tr}\big(\,\mathrm{ad}_x\,\mathrm{ad}_y\,\big) = 0$, $\quad \mathrm{Tr}\big(x\,y\big) = 0$ for all $x, y \in \mathfrak{n}$ $\;\Rightarrow\;$ $F_{\mu\nu} \cdot F^{\mu\nu} = 0$ for d.o.f. in $\mathfrak{n}$)

They eventually still could contribute to matter field Lagrangians.

But then, their Euler-Lagrange equations would be strange, wouldn't it?
(Some algebraic equations, without kinetic wave operator.)

Observation (*Acta.Phys.Polon.***B52**(2021)63):
The ordinary Dirac kinetic Lagrangian can be regarded with a $\mathrm{D}(1)\times\mathrm{U}(1)$ internal group.

$$\mathbf{L}_{\text{Dirac}}(\gamma^a,\ \Psi,\ \nabla_b\Psi) \qquad = \qquad \mathbf{v}_\gamma\ \ \mathrm{Re}\left(\overline{\Psi}\,\gamma^c\,\mathrm{i}\nabla_c\Psi\right)$$

($\mathbf{v}_\gamma$ is the metric volume form, ∇_b is the metric + $\mathrm{D}(1)\times\mathrm{U}(1)$ gauge-covariant derivation.)

This, besides usual local $\mathrm{U}(1)$ gauge invariance, is also locally $\mathrm{D}(1)$ gauge invariant:

$$\begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \end{pmatrix} \qquad \xmapsto{\Omega>0} \qquad \begin{pmatrix} \Omega^{-\frac{3}{2}}\ \Psi \\ \Omega^{-1}\ \gamma^a \\ \Omega^{-\frac{3}{2}}\ \nabla_b\,\Omega^{\frac{3}{2}} \ = \ \nabla_b + \Omega^{-\frac{3}{2}}\ \mathrm{d}_b\,\Omega^{\frac{3}{2}} \end{pmatrix},$$

But has also an unusual hidden affine “shift” symmetry:

$$\begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \end{pmatrix} \qquad \xmapsto{C_d} \qquad \begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b + C_b \end{pmatrix}$$

(C_d is any real covector field, i.e. a $\mathrm{D}(1)$ gauge potential.)

Combination of the two causes an unusual local internal symmetry:

$$\begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \end{pmatrix} \quad \stackrel{\Omega>0}{\longmapsto} \quad \begin{pmatrix} \Omega^{-\frac{3}{2}}\;\Psi \\ \Omega^{-1}\;\gamma^a \\ \nabla_b \quad + \cancel{\Omega^{-\frac{3}{2}}\,\mathrm{d}_b\,\Omega^{\frac{3}{2}}} \end{pmatrix}.$$

The $\mathrm{D}(1)$ group can act locally internally and faithfully on matter fields,
but without a compensating $\mathrm{D}(1)$ gauge field!
(The $\mathrm{D}(1)$ gauge field is formally present, but can be transformed out from the Lagrangian.)

Can similar trick be used to get rid of nilpotent gauge bosons?
Answer: **yes.**

András László, Lars Andersson, Błażej Ruba: **arXiv1909.02208** as a toy model.

Necessary condition for this: to reside in a normal sub-Lie algebra of internal symmetries.

Our concrete example group

The smallest nilpotent Lie algebra is the 3 generator Heisenberg Lie algebra $\mathfrak{h}_3$.
Generated by q, p, e, the only nonzero bracket being $[q, p] = K\, e$ (K arbitrary nonzero real).
The outer derivations of the Lie algebra of $\mathfrak{h}_3(\mathbb{C})$ is nothing but $\mathfrak{gl}(2, \mathbb{C})$.
Thus, one can form $\mathfrak{h}_3(\mathbb{C}) \inplus \mathfrak{gl}(2, \mathbb{C})$ ($\mathfrak{gl}(2, \mathbb{C})$ mixes q, p, while scales e by trace).

$\Rightarrow$

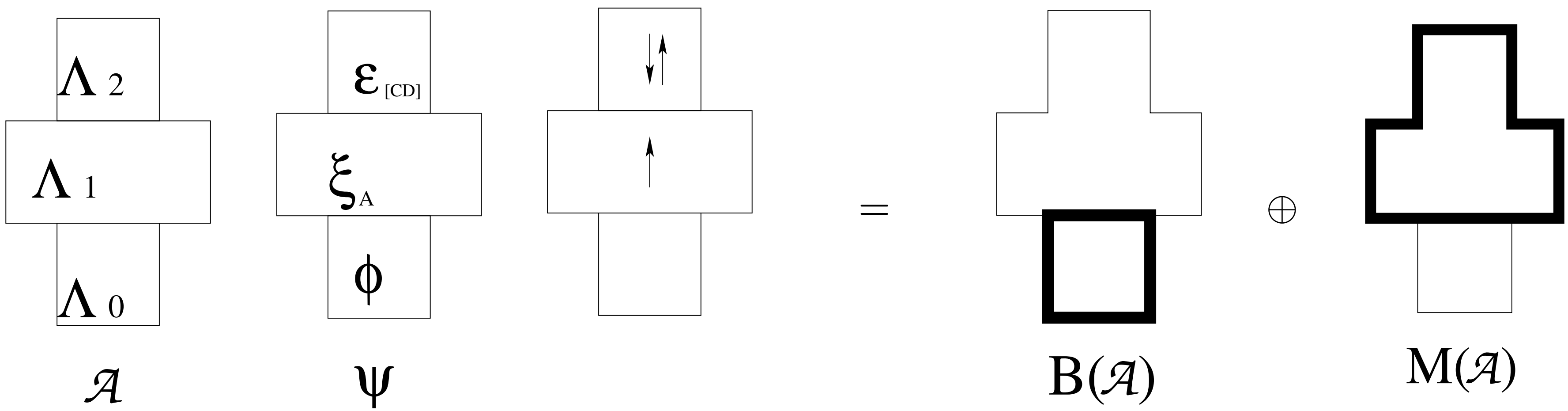
$$\Big(\underbrace{\mathfrak{t}}_{\text{translations}} \oplus \underbrace{\mathfrak{h}_3(\mathbb{C})}_{\text{Heisenberg Lie alg}} \Big) \inplus \Big(\overbrace{\underbrace{\mathfrak{u}(1)}_{\text{compact internal}} \oplus \underbrace{\mathfrak{d}(1)}_{\text{dilations}} \oplus \underbrace{\mathfrak{sl}(2, \mathbb{C})}_{\text{Lorentz}}}^{\mathfrak{gl}(2,\mathbb{C})} \Big)$$

is indecomposable conservative Poincaré Lie alg extension with a compact component.
(**`arXiv1909.02208`**)

We construct a generally covariant Lagrangian, by taking a vector bundle with
$\mathrm{H}_3(\mathbb{C}) \rtimes \mathrm{GL}(2, \mathbb{C})$
as structure group.

First, we need to find nice defining representation and its invariants.

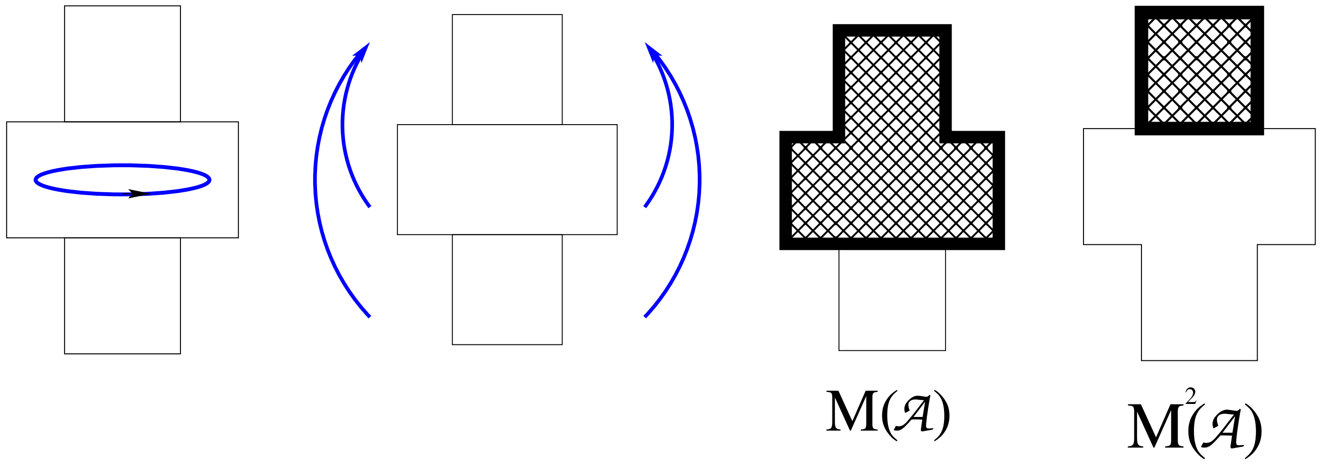
Let $\mathcal{A}$ be a 2 generator (4 dimensional) complex Grassmann algebra.
(Representation $\mathcal{A} \equiv \Lambda(S^*)$ is always possible, where S is two-spinor space.)



It turns out that $\mathfrak{h}_3(\mathbb{C})$ is isomorphic to $L_{M(\mathcal{A})}$ (left multiplication, i.e. "particle insertion").

And of course, $\mathfrak{gl}(2,\mathbb{C})$ naturally acts on $\mathcal{A} \equiv \Lambda(S^*)$ as $\mathfrak{gl}(S^*) \equiv \mathfrak{u}(1) \oplus \mathfrak{d}(1) \oplus \mathfrak{sl}(2,\mathbb{C})$.

There no invariant subspaces within $\mathcal{A}$ such that it has invariant complement.
(reducible, but still indecomposable)



(This can generally occur for representations of non-semisimple groups.)

Represent now an elemet $\psi \equiv \left(\phi,\, \xi_{A},\varepsilon_{[AB]}\right)$.

Then, the only independent $\mathrm{H}_3(\mathbb{C}) \rtimes \mathrm{SL}(2,\mathbb{C})$-invariant $\mathcal{A} \to \mathbb{C}$ function:
$\psi \mapsto b(\psi) := \phi$ (scalar component function).

The only independent $\mathrm{H}_3(\mathbb{C}) \rtimes \mathrm{SL}(2,\mathbb{C})$-invariant $\mathcal{A} \times \mathcal{A} \to \mathbb{C}$ functions:

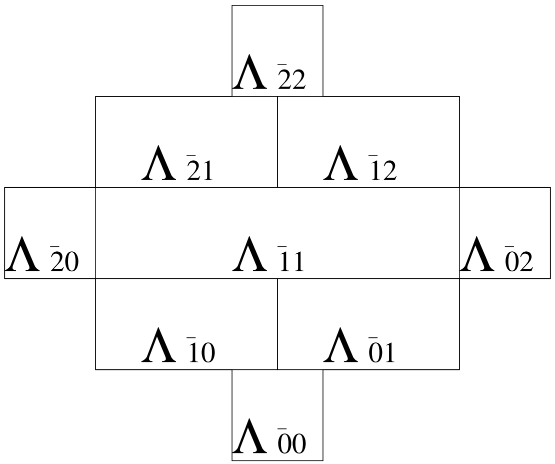
$(\psi, \psi') \mapsto b(\psi)$,

$(\psi, \psi') \mapsto b(\psi')$,

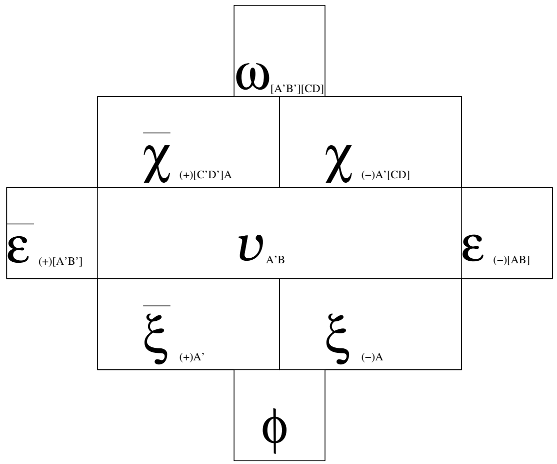
$(\psi, \psi') \mapsto \lambda(\psi,\, \psi') := -\frac{1}{2}\boldsymbol{\epsilon}^{[AB]}\left(\xi_{A}\xi'_{B} - \xi'_{A}\xi_{B} + \phi\,\varepsilon'_{[AB]} - \phi'\,\varepsilon_{[AB]}\right)$. ⟵!!!

(unique invariant symplectic form!)

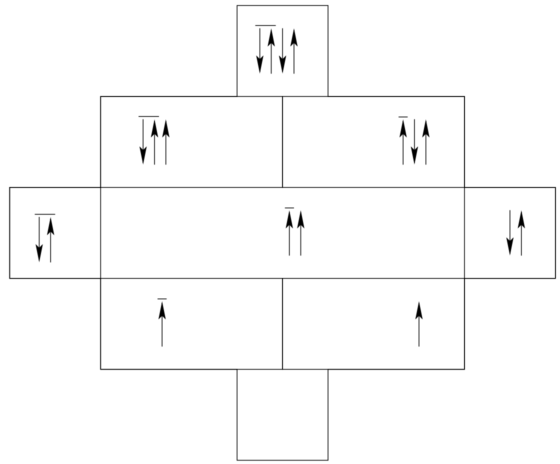
So, our group acts linearly and faithfully on $\mathcal{A}$.
Our representation space will be $A := \bar{\mathcal{A}} \otimes \mathcal{A}$, with canonical action of $\mathrm{H}_3(\mathbb{C}) \rtimes \mathrm{GL}(2, \mathbb{C})$.
(“spin algebra”, can be represented as $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$)



A

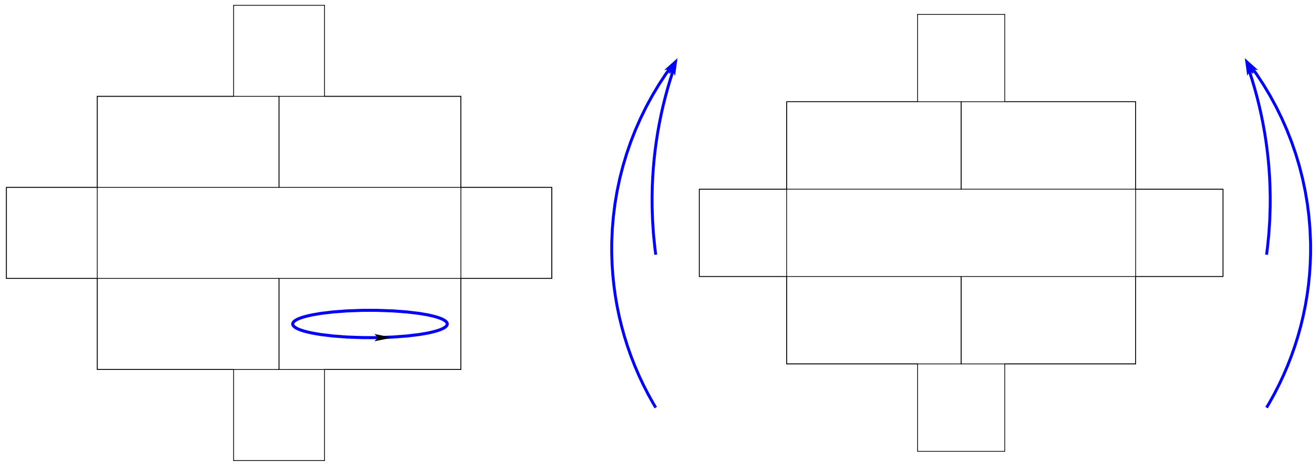


Ψ



Heuristically, it is a creation operator algebra of two fermion kinds, being charge conjugate.
(2 generating d.o.f. obeying Pauli principle + the other copy is just charge conjugate)

Again, no invariant subspace with invariant complement (indecomposable).



The only indep. $A \to \mathbb{C}$ inv. function is again the $b : \Psi \mapsto b(\Psi)$ scalar component function.

The only independent $A \times A \to \mathbb{C}$ invariants:

$(\Psi, \Psi') \mapsto b(\Psi)$,

$(\Psi, \Psi') \mapsto b(\Psi')$,

$(\Psi, \Psi') \mapsto L(\Psi, \Psi')$. $\longleftarrow$ inherits from $\mathcal{A}$ through $\bar{\lambda} \otimes \lambda$, plays the key role.

(unique invariant symmetric nondegenerate bilinear form!)

Let T be a 4 dim real vector space ("tangent space").

In ordinary two-spinor calculus the *soldering form* is a $\sigma_a^{\ A'A} : T \to \mathrm{Re}(\bar{S} \otimes S)$ linear injection.

This concept survives our generalization:

$$\sigma_a^{\ A'A} \ : \ T \longrightarrow \mathrm{Re}\Big(\underbrace{\big(M(\bar{\mathcal{A}})/M^2(\bar{\mathcal{A}})\big)^*}_{\equiv \bar{S}} \otimes \underbrace{\big(M(\mathcal{A})/M^2(\mathcal{A})\big)^*}_{\equiv S} \Big)$$

is perfectly well defined.

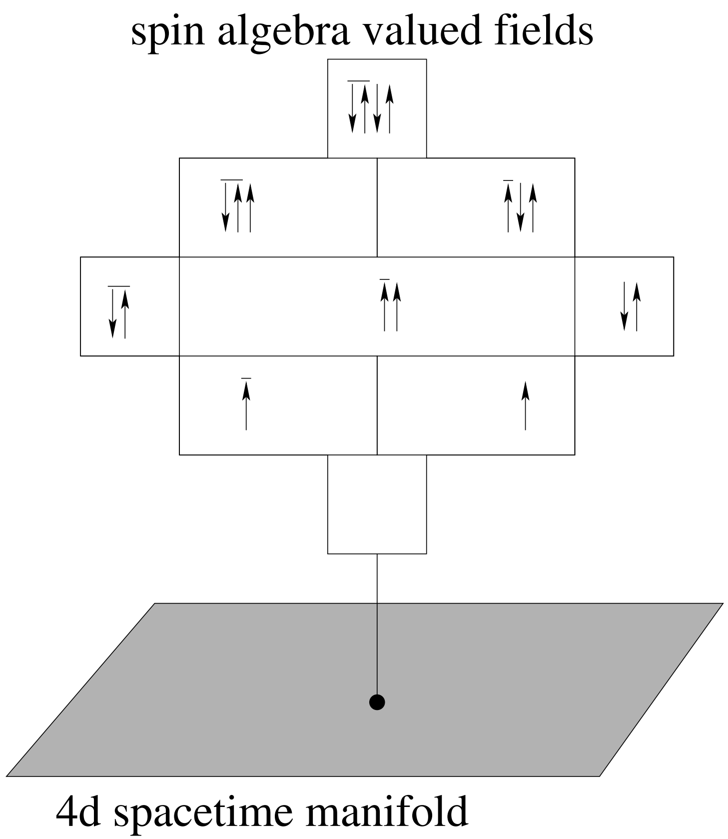
So, as usual, ($\bar{\boldsymbol{\epsilon}}_{[A'B']}\,\boldsymbol{\epsilon}_{[AB]}$ in ordinary GR)

$$g(\sigma, \boldsymbol{\omega})_{ab} \ := \ \sigma_a^{\ A'A} \ \sigma_b^{\ B'B} \ \boldsymbol{\omega}_{[A'B'][AB]}$$

is a Lorentz signature metric, where $\boldsymbol{\omega}_{[A'B'][AB]} \in M^4(A) \equiv \Lambda_{\bar{2}2} \equiv \overset{2}{\wedge}\, \bar{S}^* \otimes \overset{2}{\wedge}\, S^*$ fixed.

(Our group, by construction, respects causal structure.)

We take A-valued vector bundle over 4d spacetime manifold, with discussed structure group.
(The invariant Lagrangians are based largely on the invariant form L.)



Encodes per spacetime point:
creation op. algebra for 2 fundamental d.o.f.,
Pauli principle,
charge conjugation.

Early attempt: R.M.Wald, S.Anco: *Phys.Rev.*__D39__(1989)2297 with algebra valued fields.
(But they took too simple algebra for this.)

Lagrange form is a pointwise volume form valued map:

$$\left(o,\, \sigma_a^{\ A'A},\, \Psi,\, \nabla_a\Psi,\, F_{bc}(\nabla) \right) \quad \longmapsto \quad \mathbf{L}\left(o,\, \sigma_a^{\ A'A},\, \Psi,\, \nabla_a\Psi,\, F_{bc}(\nabla) \right)$$

(o: orientation, $F(\nabla)_{bc}$: curvature tensor)

We assume a Palatini-like variation principle, i.e. ∇_a is unconstrained.

We get the following invariant Lagrangians:
YM for the $\mathfrak{u}(1)$ gauge field,
EH-like term for the $\mathfrak{sl}(2,\mathbb{C})$ connection,
Dirac-like term for Ψ,
KG-like term for Ψ is forbidden by $\nabla_a \mapsto \nabla_a + C_a$,
fourth order self-interaction term.

(only 1 coupling factor is independent — strength of gravitation)

Could be interesting minimal toy model.

Re-understanding Coleman–Mandula (knowing Levi decomp.th.)

Essence of Coleman–Mandula-like no-go theorems (assuming finite dimensionality):

- Full symmetry group is a Poincaré group extension $\Rightarrow$ O'Raifeartaigh A or B or C. (no other possibilities exist Lie group theoretically)
- Complementing symmetries to Poincaré symmetries have positive definite invariant "internal" scalar product $\Rightarrow$ no O'Raifeartaigh B.
- No symmetry breaking present $\Rightarrow$ no O'Raifeartaigh C.

Our mechanism: internal Lie alg is $\mathfrak{n} \niplus \mathfrak{g}$ $\Rightarrow$ internal scalar product degenerates on $\mathfrak{n}$ $\Rightarrow$ $\checkmark$
SUSY: similar degeneration over the pure supertranslations q $\Rightarrow$ $\checkmark$
(super-Lie algebra can be used to say that we allow certain kind of nilradical)

Both are O'Raifeartaigh B mechanism.

Attention! Coleman-Mandula also has a hidden assumption:

- Symmetry generators strictly conserve 1-particle subspaces.
  $\Rightarrow$ No extra representation space for generators possibly stepping in the Fock hierarchy.
  (Are we sure on this assumption?!)

Summary

- **SUSY experimentally not visible at present.** As of 2021 status.
- **Mathematical alternatives to SUSY exist.** Are also O'Raifeartaigh B type, as SUSY.
- **The alternative: "conservative" extensions of the spacetime symmetries.** The complementing symmetries to spacetime symmetries are all strictly internal.
- **Example constructed.** At present, merely with $\mathrm{U}(1)$ as compact gauge group.
- **It connects gauge and spacetime symmetries.** Similarly to extended SUSY.
- **Harmonizes with present experimental situation.** Extra symmetries are "hidden".
- **Symmetry "hiding".** Symmetry breaking is not the only mechanism to get rid of exotics.
- **Testing on minimal model.** It seems we can construct a minimal model.

*J.Phys.***A50**(2017)115401, *Acta.Phys.Polon.***B52**(2021)63, `arXiv1909.02208`.

Backup

The invariant Lagrangians

Yang–Mills term.

$$\mathbf{L}_{\mathrm{YM}}\big(o,\, \sigma_a^{\ A'A},\, \Psi,\, \nabla_a\Psi,\, F(\nabla)_{ab}\big) \;=$$

$$\mathbf{v}(o,\sigma)\, g(\sigma)^{ac}\, g(\sigma)^{bd}\ \mathrm{Im}\big(\ \mathrm{Tr}|_{\Lambda_{\bar{0}1}}\ F(\nabla)_{ab}\big)\ \mathrm{Im}\big(\ \mathrm{Tr}|_{\Lambda_{\bar{0}1}}\ F(\nabla)_{cd}\big).$$

Einstein–Hilbert-like term.

$$\mathbf{L}_{\mathrm{EH}}\big(o,\, \sigma_a^{\ A'A},\, \Psi,\, \nabla_a\Psi,\, F(\nabla)_{ab}\big) \;=$$

$$\mathbf{v}(o,\sigma)\, g(\sigma)^{ab}\, L(\overline{\Psi},\Psi)\ \mathrm{Re}\big(\ \mathrm{Tr}|_{\Lambda_{\bar{0}1}}\ \big(\mathrm{i}\Sigma(\sigma)_a{}^c\, F(\nabla)_{cb}\big)\big).$$

Klein–Gordon-like term is not allowed.

$$\mathbf{L}_{\mathrm{KG}}\big(o,\, \sigma_a^{\ A'A},\, \Psi,\, \nabla_a\Psi,\, F(\nabla)_{ab}\big) \;=$$

$$\mathbf{v}(o,\sigma)\, g(\sigma)^{ab}\, L\big(\overline{\mathrm{i}\nabla_a(\Psi)},\, \mathrm{i}\nabla_b(\Psi)\big)$$

Dirac-like term.

$$\mathbf{L}_{\mathrm{D}}(o,\, \sigma_a^{\;A'A},\, \Psi,\, \nabla_a\Psi,\, F(\nabla)_{ab}) \;=$$

$$\mathbf{v}(o,\sigma)\,\tfrac{1}{\sqrt{2}}\;\mathrm{Re}\Big(\,L\big(\overline{\Psi},\;\gamma(\overline{\Psi},\Psi)^a\;\mathrm{i}\nabla_a(\Psi)\big)\,\Big)$$

where $\gamma(\overline{\Psi},\Psi)^a$ is a $T^* \to \mathrm{Lin}(A)$ pointwise linear map, defined as

$$\gamma(\overline{\Psi},\Psi')^a(\cdot) \;:=$$

$$\tfrac{1}{\sqrt{2}}\;\sigma^a_{\;A'A}\;\Big(\,(R_{\boldsymbol{\delta}^A}\,\overline{\Psi}\,)\;L\big(R_{\bar{\boldsymbol{\delta}}^{A'}}\,\Psi',\,\cdot\big)\;+\;(R_{\bar{\boldsymbol{\delta}}^{A'}}\,\overline{\Psi}\,)\;L\big(R_{\boldsymbol{\delta}^A}\,\Psi',\,\cdot\big)\,\Big).$$

($R_{\boldsymbol{\delta}^A}$ and $R_{\bar{\boldsymbol{\delta}}^{A'}}$ denote the pointwise injections $S^* \to R_{\mathcal{A}}$ and $\bar{S}^* \to R_{\bar{\mathcal{A}}}$)

Fourth order self-interaction potential.

$$\mathbf{L}_{\mathrm{V}}(o,\, \sigma_a^{\;A'A},\, \Psi,\, \nabla_a\Psi,\, F(\nabla)_{ab}) \;=$$

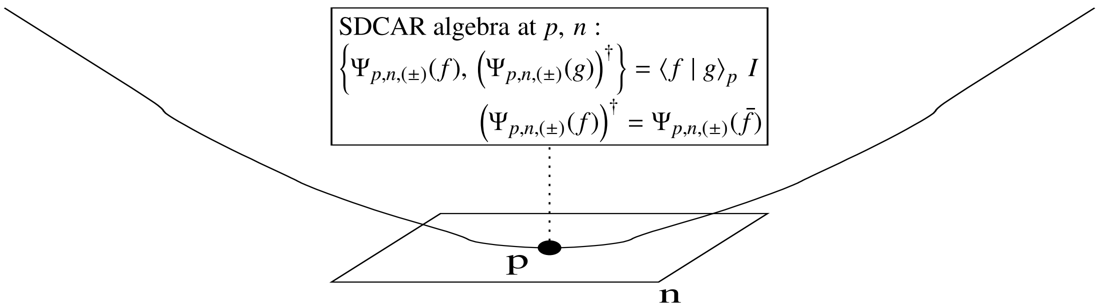
$$\mathbf{v}(o,\sigma)\,L\big(\overline{\Psi},\Psi\big)\;L\big(\overline{\Psi},\Psi\big)$$

Spin algebra and the SDCAR algebra

Let A be spin algebra.

Then, one can show that there is a correspondence:

$A \quad \longleftrightarrow \quad \mathrm{Aut}(A)$-covariant family of SDCAR algebras
(parametrized by mass shell and $\mathbb{Z} \times \mathbb{Z}$-grading of A)

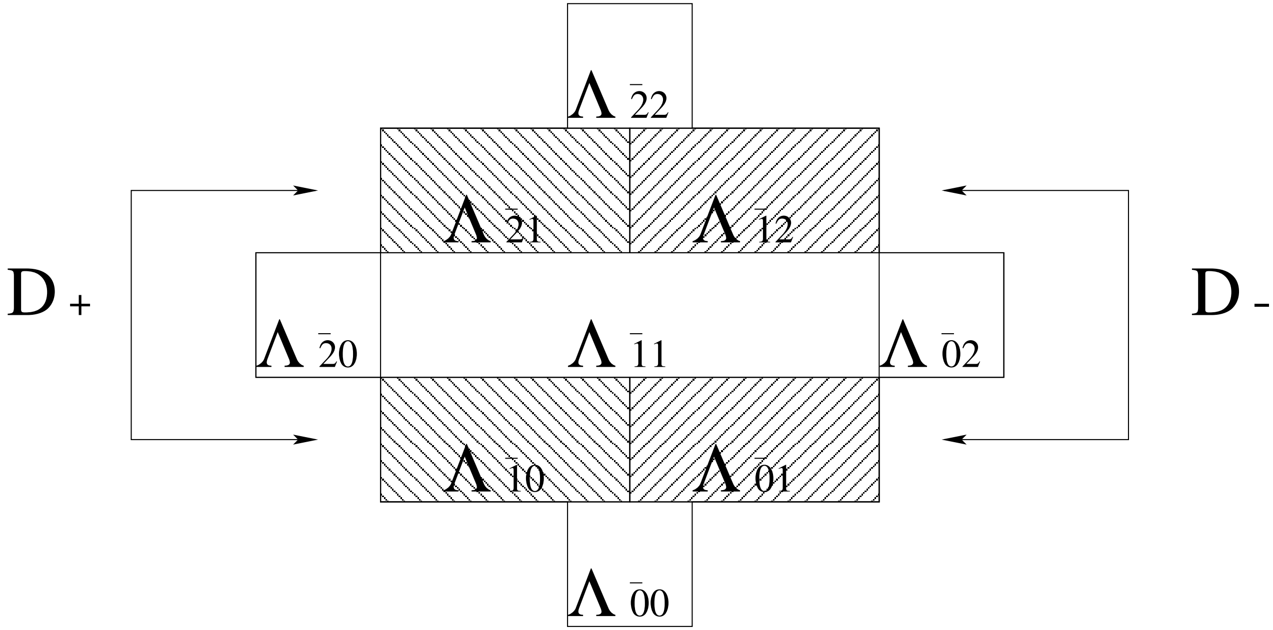


(SDCAR algebra: algebra of quantum field operators.)

So, it is really a QFT semiclassical limit.

Spin algebra and Clifford algebra

Within spin algebra, the ordinary Dirac bispinors are contained:



One can define Dirac gamma map $\gamma : \ T \rightarrow \ \mathrm{Lin}(A)$, such that it obeys Clifford relations

$$\gamma_a \gamma_b + \gamma_b \gamma_a \quad = \quad 2\,I\,g_{ab}$$

when restricted to D_+ and D_-. They have $+1$ and -1 electric charge.

One can define spin tensor Σ_{ab} implementing Lorentz generators on each $\Lambda_{\bar{p}q}$ properly, and

$$\Sigma_{ab} \quad = \quad \frac{\mathrm{i}}{2}\big(\gamma_a \gamma_b - \gamma_b \gamma_a\big)$$

holds as usual.

Spin algebra and superfields

Representation via two-spinor calculus:

Let S^* be a 2 dimensional complex vector space ("cospinor space").

Then $\Lambda(\bar{S}^*)\otimes\Lambda(S^*)$ is spin algebra.

Here $\Lambda(\mathit{some\ vector\ space})$ means exterior algebra of $\mathit{some\ vector\ space}$.

So, a spin algebra is isomorphic to ("has the structure of") $\Lambda(\bar{S}^*)\otimes\Lambda(S^*)$.
But there is a freedom in matching the canonical generators ("not natural isomorphism").

An element of $\Lambda(\bar{S}^*)\otimes\Lambda(S^*)$ consists of 9 spinorial sectors:

$$\Big(\varphi \quad \bar{\xi}_{(+)\,A'} \quad \xi_{(-)\,A} \quad \bar{\epsilon}_{(+)\,[A'B']} \quad v_{A'B} \quad \epsilon_{(-)\,[AB]} \quad \bar{\chi}_{(+)\,[C'D']A} \quad \chi_{(-)\,A'[CD]} \quad \omega_{[A'B'][CD]}\Big)$$

$\Lambda(\bar{S}^*)\otimes\Lambda(S^*)$ is similar, but $\neq$ to superfield algebra $\Lambda\big(\bar{S}^*\oplus S^*\big)$. Those would be:

$$\Big(\varphi \quad \bar{\xi}_{(+)\,A'} \quad \xi_{(-)\,A} \quad \bar{\epsilon}_{(+)\,[A'B']} \quad v_{[A'B]} \quad \epsilon_{(-)\,[AB]} \quad \bar{\chi}_{(+)\,[C'D'A]} \quad \chi_{(-)\,[A'CD]} \quad \omega_{[A'B'CD]}\Big)$$

Usual conformal invariance of Dirac Lagrangian

$$\mathbf{L}_{\text{Dirac}}(\gamma^a,\ \Psi,\ \nabla_b\Psi) \quad = \quad \mathbf{v}_\gamma\ \ \mathrm{Re}\left(\overline{\Psi}\,\gamma^c\,\mathrm{i}\nabla_c\Psi\right)$$

($\mathbf{v}_\gamma$ is the metric volume form, ∇_b is the metric + $\mathrm{U}(1)$ gauge-covariant derivation.)

Usual conformal invariance in Palatini variables:

$$\begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \end{pmatrix} \quad \xmapsto{\Omega>0} \quad \begin{pmatrix} \Omega^{-\frac{3}{2}}\ \Psi \\ \Omega^{-1}\ \gamma^a \\ \nabla_b - \frac{1}{2}(\mathrm{i}\Sigma_b{}^c - \delta_b{}^c I)(\Omega^{-1}\,\mathrm{d}_c\Omega) \end{pmatrix}.$$

(with $\Sigma_{bc} := \frac{\mathrm{i}}{2}\left(\gamma_b\gamma_c - \gamma_c\gamma_b\right)$ being the spin tensor.)

If a $\mathrm{D}(1)$ gauge field is also present in ∇_b, then also

$$\begin{pmatrix} \Psi \\ \gamma^a \\ \nabla_b \end{pmatrix} \quad \xmapsto{\Omega>0} \quad \begin{pmatrix} \Omega^{-\frac{3}{2}}\ \Psi \\ \Omega^{-1}\ \gamma^a \\ \Omega^{-\frac{3}{2}}\ \nabla_b\,\Omega^{\frac{3}{2}} \ = \ \nabla_b + \Omega^{-\frac{3}{2}}\ \mathrm{d}_b\,\Omega^{\frac{3}{2}} \end{pmatrix},$$

is a symmetry.

Discrete symmetries and unification

Relation of unital connected component and of discrete symmetries:

$$\underbrace{E}_{\text{finite dim real Lie group}} = \underbrace{E_0}_{\text{unital connected component}} \cdot \underbrace{D}_{\text{discrete symmetries}}$$

Can be very tricky, not necessarily a semi-direct product. (May have a role in unification?)

Sometimes semi-direct product, like Lorentz group:

$$\underbrace{\mathrm{O}(1,3)}_{\text{all Lorentz transformations}} = \underbrace{\mathrm{SO}^{\uparrow}(1,3)}_{\text{unital connected component}} \rtimes \underbrace{\{\mathrm{identity},\, p,\, t,\, pt \text{ transformations}\}}_{\text{discrete symmetries (they do close as standalone group)}}$$

But can be more entangled, like Clifford automorphisms of (1,3) signature spacetime:

$$\underbrace{\mathrm{Aut}\big(\mathrm{Cl}(1,3)\big)}_{\text{all Clifford automorphisms}} = \underbrace{\mathrm{Aut}_0\big(\mathrm{Cl}(1,3)\big)}_{\text{unital connected component}} \cdot \underbrace{\{I,\, C,\, P,\, T,\, CP,\, CT,\, PT,\, CPT \text{ operators}\}}_{\text{discrete symmetries (does not close as standalone group)}}$$

because $PTP^{-1} = (-I)\,T,\ TPT^{-1} = (-I)\,P$ etc, and $-I$ sits in $\mathrm{Aut}_0\big(\mathrm{Cl}(1,3)\big)$.