QCD transition in magnetic fields

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Outline - first part

- introduction
  - strong interactions at finite temperature
  - quark-gluon plasma exposed to magnetic fields
  - appetizer: chiral magnetic effect in heavy-ion collisions
  - approaches to study QCD
- free case: energy levels
  - non-relativistic case, infinite volume
  - relativistic case, infinite volume
  - relativistic case, on the torus
  - Hofstadter's butterfly
- free case, thermodynamic potential
  - representation at finite T with Matsubara frequencies
  - treatment via Mellin transformation
  - alternative derivation: Schwinger proper-time method
  - alternative representation: with energies
  - charge renormalization vs B-dependent divergences
  - observables derived from $\log \mathcal{Z}$

Outline - second part

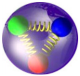
- numerical results I: phase diagram
  - symmetries and order parameters
  - predictions from effective theories and models
  - magnetic catalysis and inverse catalysis
  - transition temperature, nature of transition at nonzero B
- numerical results II: equation of state
  - concept of the pressure in magnetic fields
  - magnetization, magnetic susceptibility
  - comparison to hadron resonance gas model
  - squeezing-effect in heavy-ion collisions
- numerical results III: chiral magnetic effect
  - electric polarization of CP-odd domains
  - comparison to model predictions

Literature

- Landau-Lifshitz Vol.3 Quantum mechanics, chapter XV. (non-relativistic eigenvalue problem)
- Akhiezer, Berestetskii: Quantum electrodynamics, chapter 12. (relativistic eigenvalue problem)
- Kapusta: Finite-temperature field theory, chapter 2. (functional integral for fermions/bosons)
- Al-Hashimi, Wiese: "Discrete accidental symmetry for a particle in a constant magnetic field on a torus"
- Hofstadter: "Energy levels and wavefunctions of Bloch electrons in rational and irrational magnetic fields"
- Schwinger: "On gauge invariance and vacuum polarization"
- Dunne: "Heisenberg-Euler effective Lagrangians: basics and extensions"

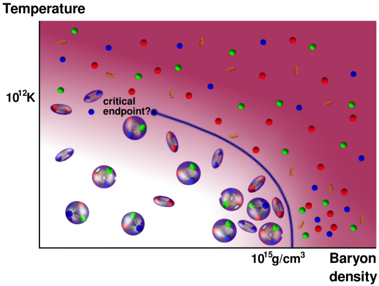
QCD and quark-gluon plasma

- elementary particle interactions:
  gravitational, $\underbrace{\text{electromagnetic, weak, strong}}_{\text{Standard Model}}$
- strong sector: Quantum Chromodynamics
- elementary particles: quarks ($\sim$ electrons) and gluons ($\sim$ photons)
  but: they cannot be observed directly
  $\Rightarrow$ *confinement* at low temperatures
- asymptotic freedom [Gross, Politzer, Wilczek '04]
  $\Rightarrow$ heating or compressing the system leads to *deconfinement*: quark-gluon plasma is formed
- transition between the two phases
  characteristics: order (1st/2nd/crossover)
  critical temperature T_c
  equation of state



QCD phase diagram

- why is the physics of the quark-gluon plasma interesting?
  - large T: early Universe, cosmological models
  - large ρ: neutron stars
  - large T and/or ρ: heavy-ion collisions, experiment design



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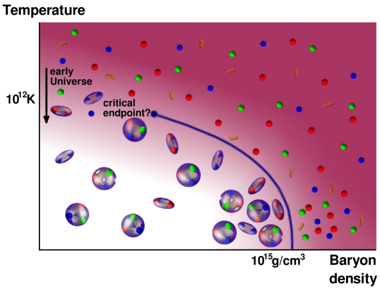

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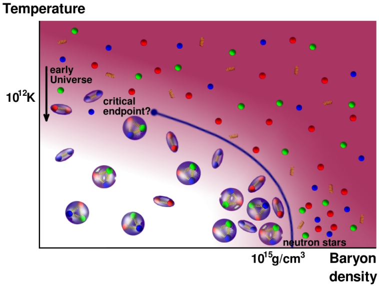

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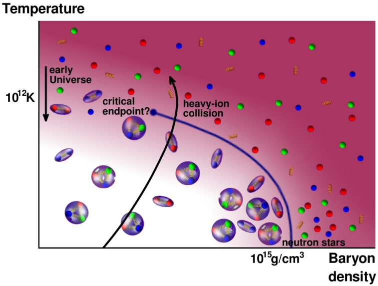

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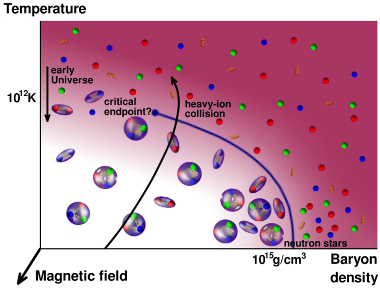
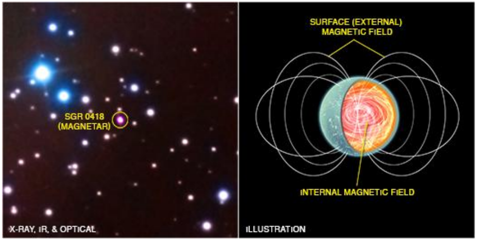


- additional, relevant parameter:
  - external magnetic field B

Example 1: neutron star



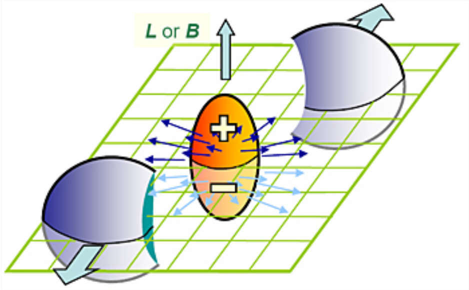
[Rea et al. '13]

- possible quark core at center with high density, low temperature
- magnetars: extreme strong magnetic fields

Typical magnetic fields

• magnetic field of Earth	10^{-5} T
• common magnet	10^{-3} T
• strongest human-made field in lab	10^{2} T
• magnetar surface	10^{10} T
• magnetar core	?

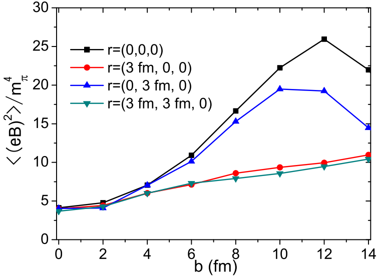
Example 2: heavy-ion collision



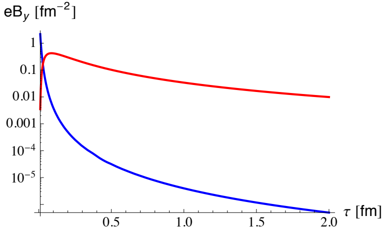
[STAR collaboration, '10]

- off-central collisions generate magnetic fields:
  strength controlled by $\sqrt{s}$ and impact parameter (centrality)
- strong (but very uncertain) time-dependence
- anisotropic spatial gradients

Example 2: heavy-ion collision



[Bloczynski et al. '12]



[Gursoy et al '13]

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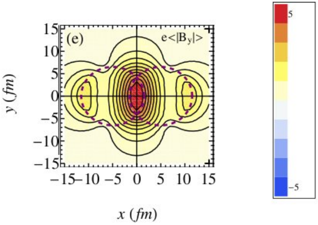
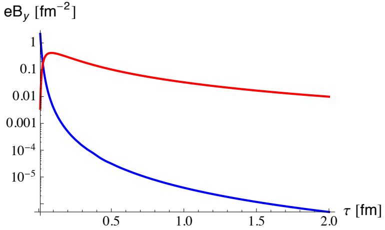


[Deng et al '12]



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• magnetar core	?
• LHC Pb-Pb at 2.7 TeV, $b = 10$ fm [Skokov '09]	10^{15} T

convert: 10^{15} T $\approx 10 m_\pi^2 \approx 2\Lambda_{\mathrm{QCD}}^2$

$\Rightarrow$ electromagnetic and strong interactions can compete

Chiral magnetic effect

- QCD is parity-symmetric (neutron EDM $< 10^{-26} e$ cm)

$$\mathcal{L}_{\rm QCD} = \sum_f \bar{\psi}_f \left(\not{D} + m_f \right) \psi_f + \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu} + \theta \cdot \underbrace{\frac{1}{16\pi^2} \mathrm{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu}}_{Q_{\rm top}}$$

$\Rightarrow \theta < 10^{-10}$ (strong CP problem)

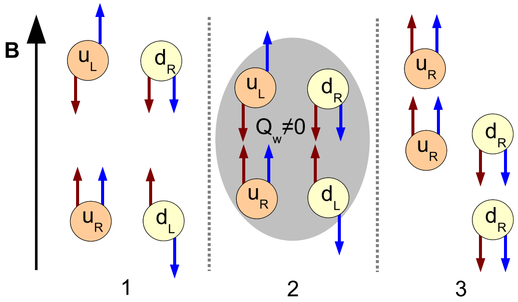
- axial anomaly

$$N_R - N_L \equiv \int d^4x \, \partial_\mu j^{\mu 5} = 2 Q_{\rm top}$$

$\Rightarrow$ topology converts between left- and right-handed quarks

Chiral magnetic effect

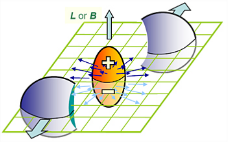
- local CP-violation through domains with $Q_{\rm top} \neq 0$?
- detect them through magnetic field B [Kharzeev et al. '08]

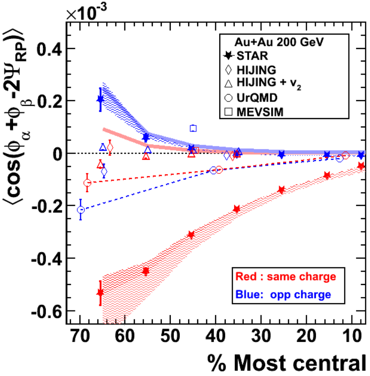


1. quarks interact with B: spins aligned
2. quarks interact with topology: chiralities (helicities) "aligned"
3. result: charge separation

Chiral magnetic effect

- $Q_{\rm top}$-domains fluctuate, direction of B fluctuates
  $\Rightarrow$ effect vanishes on average
- correlations may survive $(\alpha, \beta = \pm)$
  $a_{\alpha\beta} = -\cos\left[(\Phi_\alpha - \Psi_{\rm RP}) + (\Phi_\beta - \Psi_{\rm RP})\right]$





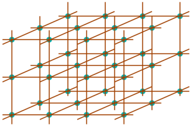
[STAR collaboration '09]

- need 3-particle correlations (technically complicated)
- CME prediction:
  $a_{++} = a_{--} = -a_{+-} > 0$
- CP-even backgrounds should be subtracted

Approaches to study QCD

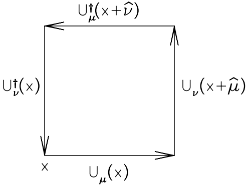
- various methods in various regimes:
  - high T/B: perturbation theory
  - low T/B: chiral perturbation theory, hadronic models
  - transition region: non-perturbative methods, lattice gauge theory [Wilson, '74]

- discretize quark and gluon fields ψ and A_μ on a 4D space-time lattice with spacing a



  - use $U_\mu = e^{iaA_\mu}$ instead of A_μ
  - U_μ: links, ψ: sites

- example: gauge action $F_{\mu\nu}F_{\mu\nu}(x) \sim$



Lattice simulations

- functional integral

$$\mathcal{Z} = \int \mathcal{D}U_\mu \, \mathcal{D}\bar{\psi} \, \mathcal{D}\psi \; \mathsf{exp}\left(- \int \mathsf{d}^4x \, \mathcal{L}_{\mathrm{QCD}} \right)$$

Lattice simulations

- functional integral

$$\mathcal{Z} = \int \mathcal{D}U_\mu \, \exp\left(-\int \mathrm{d}^4x \, \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu} \right) \cdot \prod_f \det\left(\not{D} + m_f^{\mathrm{lat}} \right)$$

Lattice simulations

- functional integral

$$\mathcal{Z} = \int \mathcal{D}U_\mu \, \exp\left(- \int \mathrm{d}^4x \, \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu} \right) \cdot \prod_f \det\left(\not{D} + m_f^{\mathrm{lat}} \right)$$

- $\mathcal{Z}$ analogous to partition function of a 4D statistical physics system; temperature and volume given as

$$T = 1/(N_t a), \qquad V = (N_s a)^3$$

- continuum limit with T and V fix:

$$a \to 0 \quad \leftrightarrow \quad N_s, N_t \to \infty, N_s/N_t = \mathrm{fix}$$

- $\mathcal{Z}$ becomes a $\sim 10^9$ dimensional integral
  - importance sampling with weight e^{-S}
  - Monte-Carlo methods

Lattice simulations

- besides continuum limit, the biggest challenge is to simulate at the physical point: set m_f^{lat} such that the measured $m_\pi, m_p, m_\rho, \ldots$ are the same as in nature
- typical computational requirement $\mathcal{O}(10\ \mathrm{Tflop/s} \times \mathrm{year})$

$\mathcal{O}(40$ mio. core hours$)$

$\mathcal{O}(100\ \mathrm{GPU} \times \mathrm{year})$

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  - observables derived from $\log \mathcal{Z}$

Energy eigenvalues

- non-relativistic case, infinite volume

$$E_n = \frac{p_z^2}{2m} + 2|qB|(n + 1/2 - \sigma_z), \qquad \#_n = \infty$$

- relativistic case, infinite volume

$$E_n = \sqrt{p_z^2 + m^2 + 2|qB|(n + 1/2 - \sigma_z)}, \qquad \#_n = \infty$$

- relativistic case, finite volume (torus)

$$E_n = \sqrt{p_z^2 + m^2 + 2|qB|(n + 1/2 - \sigma_z)}, \qquad \#_n = \frac{|qB| \cdot L^2}{2\pi}$$

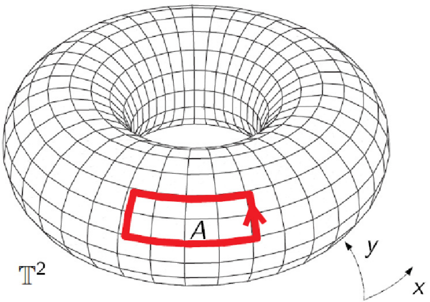
Magnetic flux quantization

- finite volume, continuum:

$$L^2 \cdot qB = \Phi = 2\pi N_b, \qquad N_b \in \mathbb{Z}$$

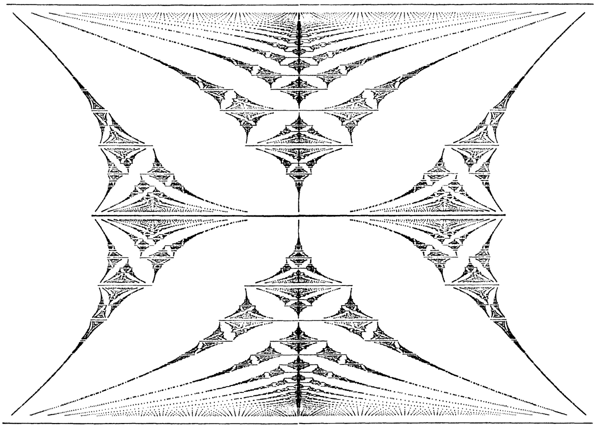
- finite volume, lattice:

$$(N_s a)^2 \cdot qB = \Phi = 2\pi N_b, \qquad 0 < N_b < N_s^2$$

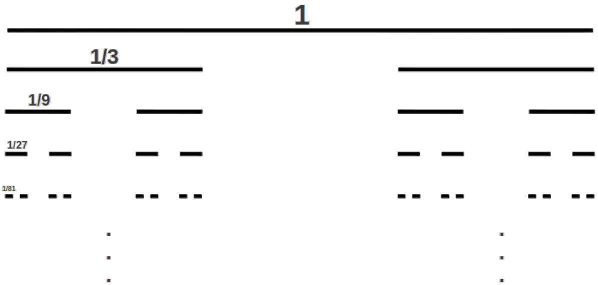


picture from [D'Elia et al '11]

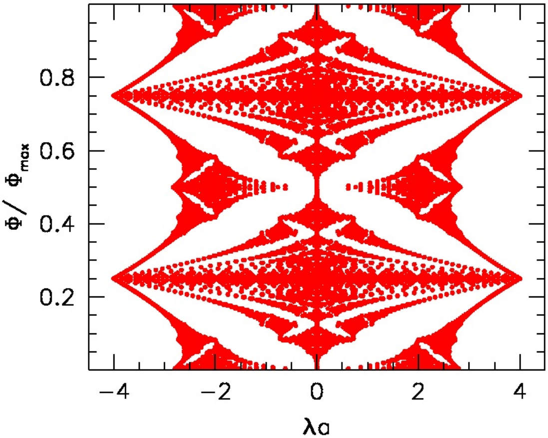
Hofstadter's butterfly



Hofstadter’s butterfly and the Cantor set

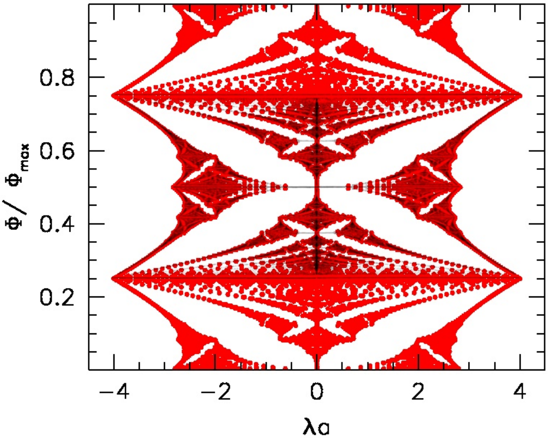


Hofstadter's cocoon on the lattice



- lattice flux quantized: $(N_s a)^2 qB = \Phi = 2\pi N_b$
  - infinite volume limit releases the butterfly
  - continuum limit kills the butterfly

Hofstadter's cocoon on the lattice



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Free energies

- charged spin$-1/2$ particle

$$f^{(1/2)}(T,B) = +\frac{1}{8\pi^2}\int \frac{\mathrm{d}s}{s^3} e^{-m^2 s} \cdot \frac{qBs \cdot \cosh(qBs)}{\sinh(qBs)} \cdot \Theta_3\left[\frac{\pi}{2}, e^{-1/(4sT^2)}\right]$$

- charged spin-0 particle

$$f^{(0)}(T,B) = -\frac{1}{8\pi^2}\int \frac{\mathrm{d}s}{s^3} e^{-m^2 s} \cdot \frac{qBs \cdot 1}{\sinh(qBs)} \cdot \Theta_3\left[0, e^{-1/(4sT^2)}\right]$$

- in general, for spin$-\sigma$:

$$f^{(\sigma)}(T,B) = (-1)^\sigma \frac{qB}{2\pi} \sum_n \sum_{\sigma_z=-\sigma}^{\sigma} \int \frac{\mathrm{d}p_z}{2\pi} \left[E_n + 2T\log\left(1+e^{-E_n/T}\right)\right]$$

$$E_n = \sqrt{p_z^2 + m^2 + 2qB(n+1/2-\sigma_z)}$$

Renormalization at zero temperature

- calculate change in f: subtract $B = 0$ contribution
- charge renormalization at $\mathcal{O}(B^2)$

$$\Delta f^{(1/2)}(0,B) = \frac{B^2}{2} + \frac{qB}{8\pi^2}\int \frac{\mathrm{d}s}{s^2} e^{-m^2 s}\cdot\left[\coth(qBs) - \frac{1}{qBs}\right]$$

$$= \frac{B_r^2}{2} + \frac{qB}{8\pi^2}\int \frac{\mathrm{d}s}{s^2} e^{-m^2 s}\cdot\left[\coth(qBs) - \frac{1}{qBs} - \frac{qBs}{3}\right]$$

- wave-function renormalization

$$B^2 = Z_q^{(1/2)} B_r^2, \qquad Z_q^{(1/2)} = 1 + q_r^2 \cdot \beta_1^{(1/2)} \cdot \log\left(\frac{m^2}{\Lambda^2}\right)$$

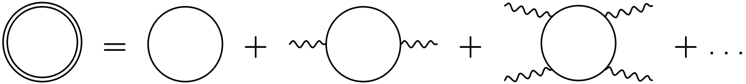
- Ward identity

$$qB = q_r B_r, \qquad q^2 = \frac{1}{Z_q^{(1/2)}} q_r^2$$

Renormalization

- expansion in the external field B diagrammatically
- $\mathcal{O}(B^0)$ contains B-independent divergences
- $\mathcal{O}(B^4)$ term is finite

- in the free case (1-loop)



- $\mathcal{O}(B^2)$ term $\propto q^2 \cdot \beta_1$

Renormalization

- expansion in the external field B diagrammatically
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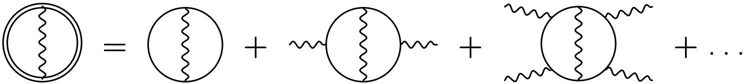
- with an internal photon to 2-loop



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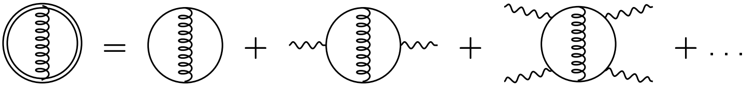

- $\mathcal{O}(B^2)$ term $\propto q^2 \cdot g^2 \cdot \beta_{1,1}$

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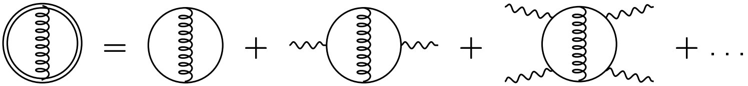

- $\mathcal{O}(B^2)$ term $\propto q^2 \cdot g^2 \cdot \beta_{1,1}$

- the coefficient of $\mathcal{O}(B^2)$ term equals the QED β-function (with QCD corrections)
  $\Rightarrow$ background field method [Abbott '81]

Renormalization – summary

- even though B is very similar to a chemical potential, it undergoes wavefunction renormalization
- B-dependent divergence $\propto (qB)^2 \beta \log(\Lambda)$, which redefines the pure magnetic energy $B_r^2/2$
- implication: susceptibility χ_B vanishes at zero T
- in the free case, UV ($\Lambda \to \infty$) and IR ($m \to 0$) divergences are intertwined
  $\Rightarrow$ quarks in SB limit are paramagnetic

$$\chi_B \propto \beta_1 \log(T/m) > 0$$

$\Rightarrow$ magnetic catalysis of the quark condensate at $T = 0$

$$\mathcal{O}((qB)^2): \quad \Delta\bar{\psi}\psi \propto \beta_1 > 0$$

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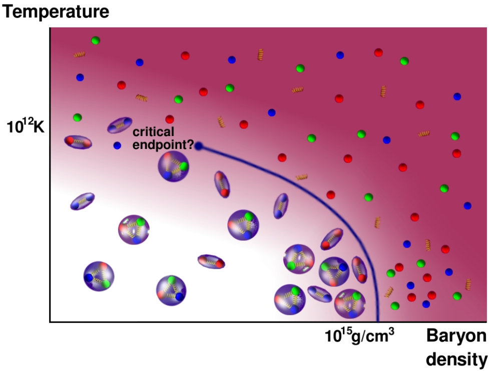
QCD phase diagram

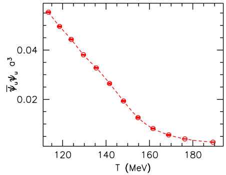


- how to map out the transition line?

Observables sensitive to transition

- chiral condensate
  $\rightarrow$ chiral symmetry breaking $m = 0$

$$\bar{\psi}_f \psi_f = \frac{\partial \log \mathcal{Z}}{\partial m_f}$$



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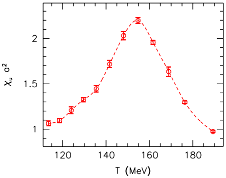
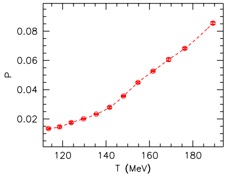


- Polyakov loop
  $\rightarrow$ deconfinement "$m = \infty$"

$$P = \mathrm{Tr}\,\exp\left[\int A_4(x,t)\,\mathrm{d}t\right]$$



Magnetic catalysis

- what happens to $\bar\psi\psi$ ($\langle +q\uparrow, -q\downarrow\rangle$) in magnetic field?
  $\Rightarrow$ magnetic moments parallel, energetically favored state
  (cf. Cooper-pairs in superconductors: Meissner effect)
- dimensional reduction $3+1 \to 1+1$ in the LLL

$$E_{LLL} = \sqrt{p_z^2 + m^2}, \qquad s_{z,LLL} = +1/2, \qquad \#_{LLL} = \frac{|qB|\cdot L_x L_y}{2\pi}$$

- chiral condensate $\leftrightarrow$ spectral density around 0 [Banks, Casher '80]

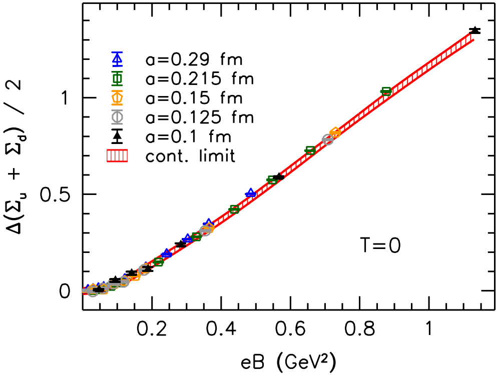
$$\bar\psi\psi \propto \rho(0)$$

- in the chiral limit, to maintain $\bar\psi\psi > 0$ (NJL [Gusynin et al '96])

$B = 0$	$\rho(p) \sim p^2 \mathrm{d}p$	"we need a strong interaction"
$B \gg m^2$	$\rho(p) \sim qB \mathrm{d}p$	"the weakest interaction suffices"

Magnetic catalysis – zero temperature

- MC at zero temperature is a robust concept: χPT, NJL, AdS-CFT, linear σ, lattice QCD at physical/unphysical m_π, ...



lattice QCD, physical m_π, continuum limit

[Bali,Bruckmann,Endrődi,Fodor,Katz,Schäfer '12]

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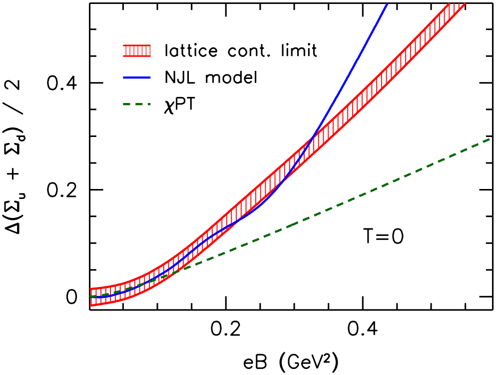

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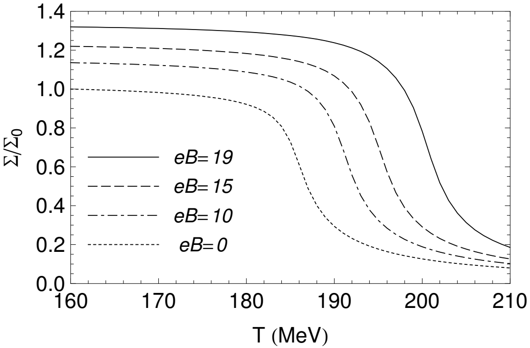

PNJL model [Gatto, Ruggieri '11]

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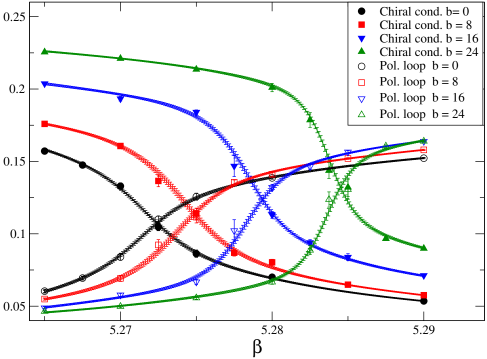

lattice QCD, unphysical m_π, coarse lattice [D'Elia et al '10]

Inverse magnetic catalysis

- lattice QCD, physical m_π, continuum limit
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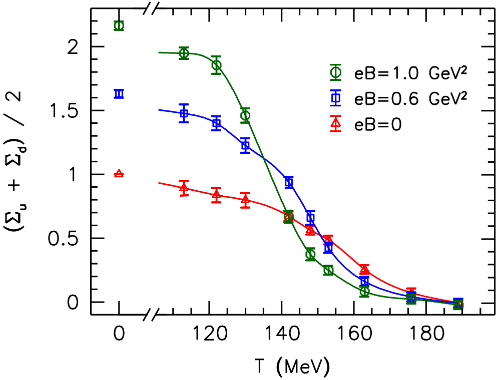

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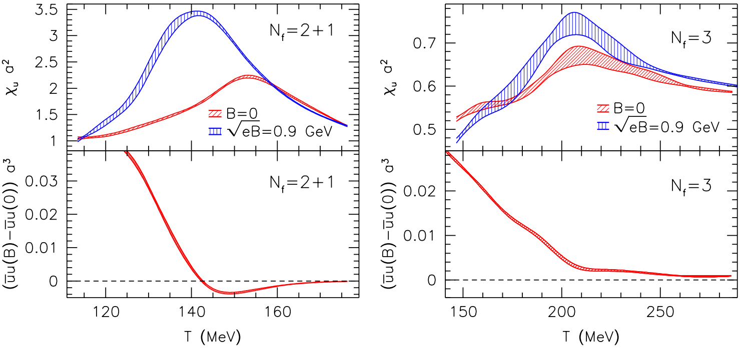
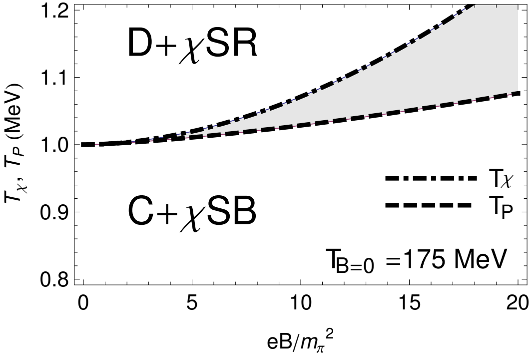


- IMC disappears if m_π is increased
  $\Rightarrow \exists\, m_\pi^\star$ such that only $m_\pi < m_\pi^\star$ gives IMC

Phase diagram

- inflection point of $\bar{\psi}\psi(T)$ defines T_c
- sinificant difference whether IMC is exhibited or not:



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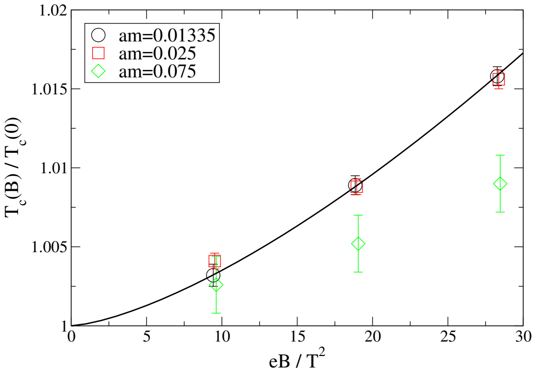

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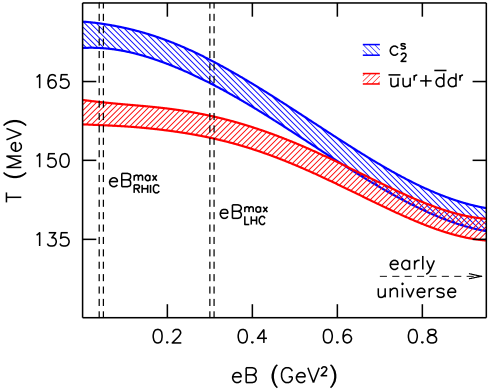

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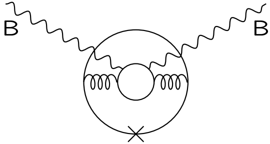
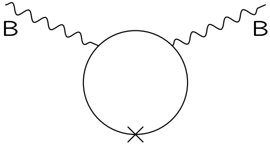
Mechanism behind IMC

- two competing mechanisms at finite B
  [Bruckmann,Endrődi,Kovács '13]
  - direct (valence) effect $B \leftrightarrow q_f$
  - indirect (sea) effect $B \leftrightarrow q_f \leftrightarrow g$

$$\bar{\psi}\psi(B) \propto \int \mathcal{D}U\, e^{-S_g} \underbrace{\det(\not{D}(B,U)+m)}_{\text{sea}} \underbrace{\mathrm{Tr}\left[(\not{D}(B,U)+m)^{-1}\right]}_{\text{valence}}$$





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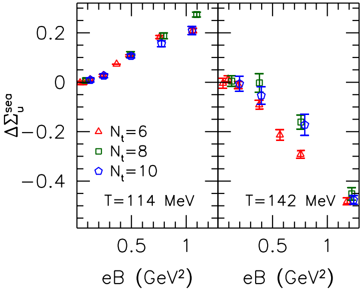
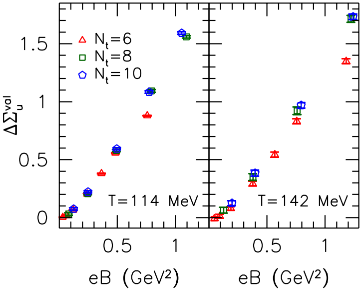
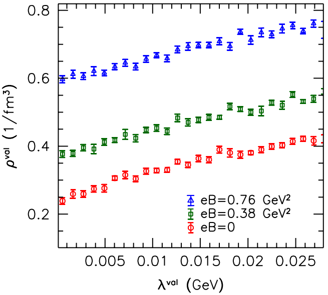




Mechanism behind IMC

- valence sector: driven by the low eigenvalues of $\not{D}$

$$\bar\psi\psi(B) \propto \int \mathcal{D}U\, e^{-S_g} \prod_i (\lambda_i^2(0) + m^2) \sum_j \frac{m}{\lambda_j^2(B) + m^2}$$



- valence sector: B creates many low eigenvalues through Landau-level degeneracy

Mechanism behind IMC

- sea sector: disfavors low eigenvalues of $\not{D}$ through det

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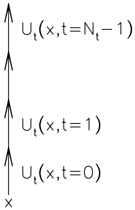

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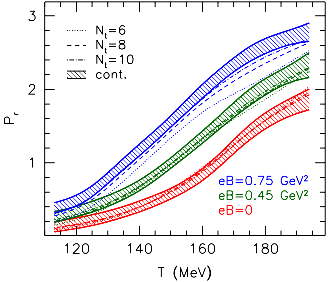

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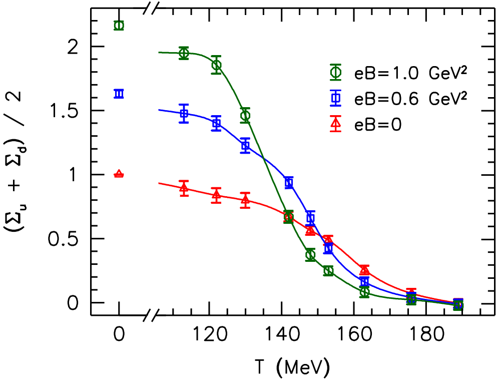
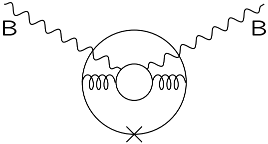
Phase diagram – conclusions

- valence and sea effects compete and around T_c the sea wins



Phase diagram – conclusions

- valence and sea effects compete and around T_c the sea wins
- lessons learned:
  - LL-picture not applicable to non-perturbative QCD
  - inclusion of dynamical quarks necessary in the models to reproduce the real phase diagram



  - important to improve effective theories/models (at $\mu_B > 0$ the lattice fails, for example)

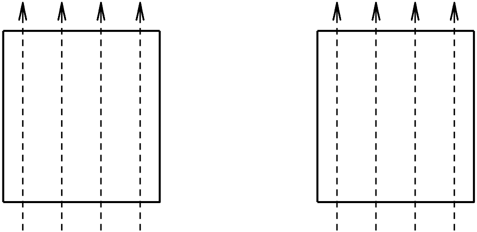
Outline - second part

- numerical results I: phase diagram
  - symmetries and order parameters
  - predictions from effective theories and models
  - magnetic catalysis and inverse catalysis
  - transition temperature, nature of transition at nonzero B
- numerical results II: equation of state
  - concept of the pressure in magnetic fields
  - magnetization, magnetic susceptibility
  - comparison to hadron resonance gas model
  - squeezing-effect in heavy-ion collisions
- numerical results III: chiral magnetic effect
  - electric polarization of CP-odd domains
  - comparison to model predictions

Concept of pressure at nonzero B

- free energy $\mathcal{F} = -T\log\mathcal{Z}$
- finite volume $V = L_x L_y L_z$, traversed by flux $\Phi = eBL_xL_y$

$$p_i = -\frac{1}{V} L_i \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}L_i}, \qquad M = -\frac{1}{V}\frac{1}{e}\frac{\partial\mathcal{F}}{\partial B}$$



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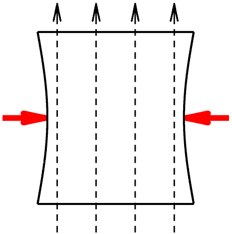
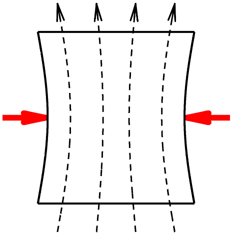


$eB = \text{fix}$



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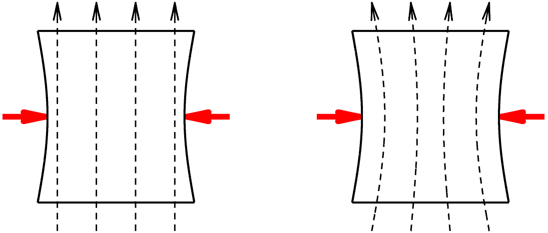


$eB = \text{fix}$ $\qquad$ $\Phi = eB \cdot L_x L_y = \text{fix}$

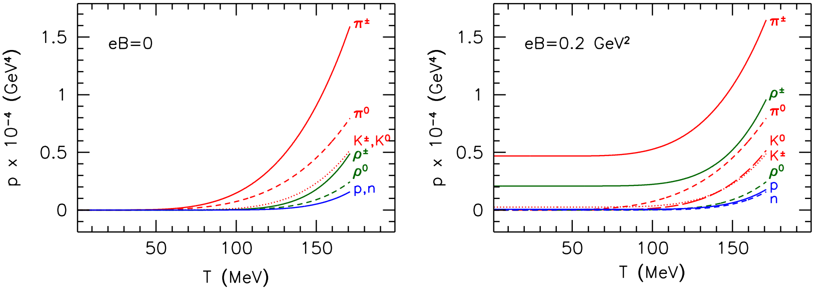
- B-scheme: $p_{x,y}^{(B)} = p_z^{(B)}$, $\quad$ Φ-scheme: $p_{x,y}^{(\Phi)} = p_z^{(\Phi)} - M \cdot eB$

Magnetization from HRG

- hadron resonance gas model: approximate free energy as

$$\mathcal{F} = \sum_h d_h \cdot \mathcal{F}_h^{\mathrm{free}}(m_h, q_h, \sigma_h)$$

QCD input: masses, charges and spins of hadrons

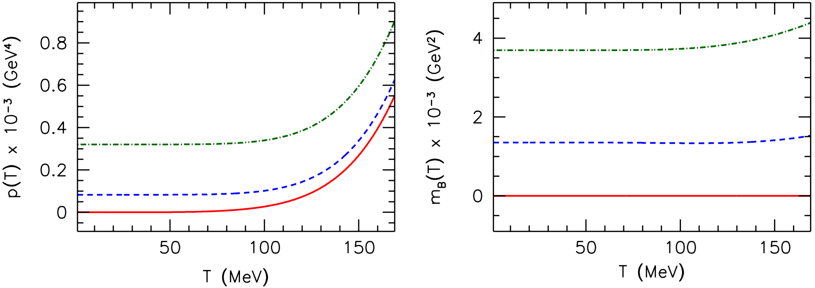


[Endrődi '13]

Magnetization from HRG

- equation of state – observables

$$p_z = -\frac{\mathcal{F}}{V}, \qquad M = -\frac{1}{V}\frac{\partial \mathcal{F}}{\partial (eB)}, \qquad s = -\frac{1}{V}\frac{\partial \mathcal{F}}{\partial T}$$



- $M > 0$: QCD vacuum is paramagnetic
- zero-T contribution is a purely quantum effect, it is “created by virtual particles”

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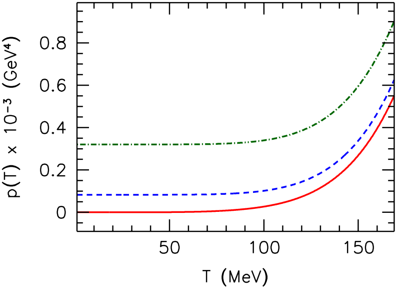

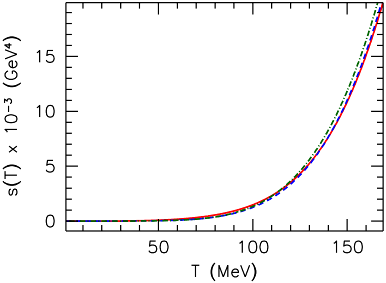

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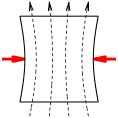
Magnetization from the lattice

- problem with $M \sim \partial\mathcal{F}/\partial B$: magnetic flux is quantized

$$\Phi = qB \cdot L_x L_y = 2\pi \cdot N_b, \qquad N_b \in \mathbb{Z}$$

$\Rightarrow$ B-derivative ill defined
$\Rightarrow$ naturally corresponds to the Φ-scheme



- magnetization determined from [Bali, Bruckmann, Endrődi et al '13]

$$p_x - p_z = -M \cdot eB$$

take an anisotropic lattice $\xi = a/a_\alpha$ [Karsch '82]

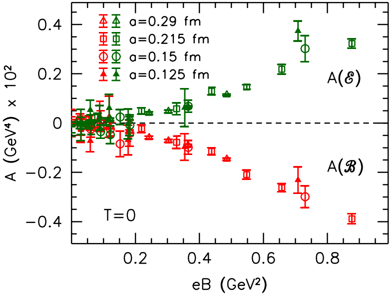
$$p_\alpha = -\xi^2 \frac{T}{V} \left.\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\xi}\right|_a$$

- p_α contains certain components of the QCD action
  $\Rightarrow$ $M \cdot eB$ contains *anisotropies* of the action

Magnetization from the lattice

- anisotropy induced in the gluonic action

$$A(\mathcal{E}) = \frac{1}{2}\left(\mathrm{tr}\mathcal{E}_x^2 + \mathrm{tr}\mathcal{E}_x^2\right) - \mathrm{tr}\mathcal{E}_z^2, \qquad A(\mathcal{B}) = \frac{1}{2}\left(\mathrm{tr}\mathcal{B}_x^2 + \mathrm{tr}\mathcal{B}_x^2\right) - \mathrm{tr}\mathcal{B}_z^2$$

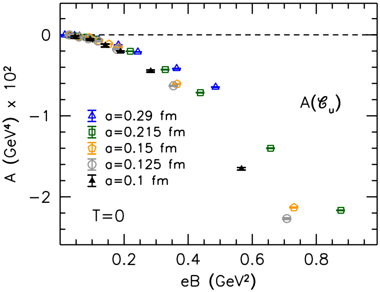


- anisotropy renormalization coefficients also enter here. . .

Magnetization from the lattice

- dominant contribution is the fermionic anisotropy

$$M \cdot eB \approx \sum_f A(\mathcal{C}_f) = \sum_f \left[\frac{1}{2} \left(\bar\psi_f \not{D}_x \psi_f + \bar\psi_f \not{D}_y \psi_f \right) - \bar\psi_f \not{D}_z \psi_f \right]$$



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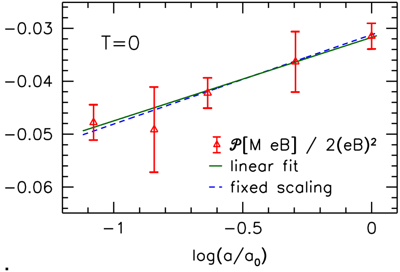


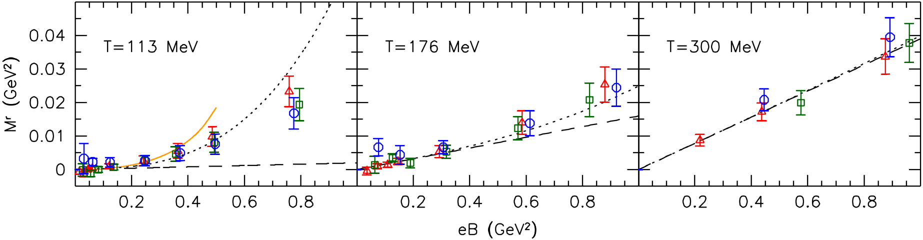
- renormalization

$$M \cdot eB = M^r \cdot eB + 2\beta_1^{\mathrm{QCD}} (eB)^2 \log a,$$

$$\beta_1^{\mathrm{QCD}} = \beta_1 \sum_f \left(\frac{q_f}{e} \right)^2 + \Delta^{\mathrm{QCD}}$$

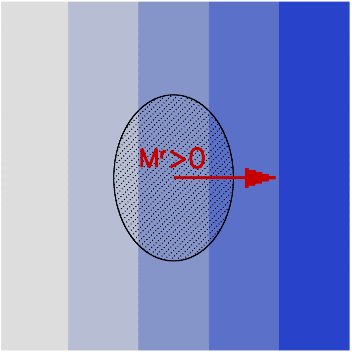
Renormalized magnetization

- subtract $\mathcal{O}((eB)^2)$ term determined at $T = 0$



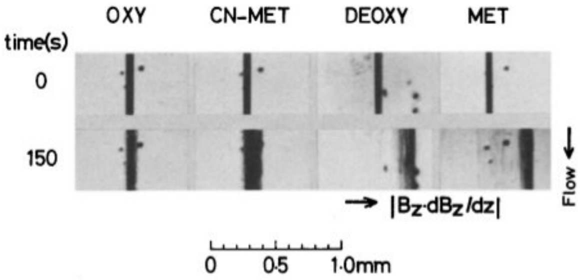
- QCD vacuum is a paramagnet
- compare to hadron resonance gas model at low T
- linear response $M^r = \chi_1 \cdot eB$ gets stronger above T_c

Paramagnetism and inhomogeneous fields



- $-\partial\mathcal{F}^r/\partial(eB) = M^r > 0$
  $\Rightarrow$ free energy $\mathcal{F}^r$ minimized in the region where B is maximal

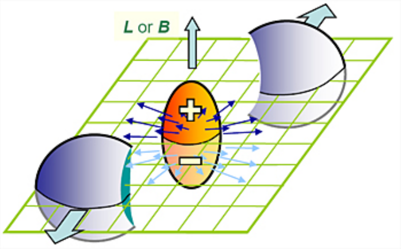
Paramagnetism - blood cells

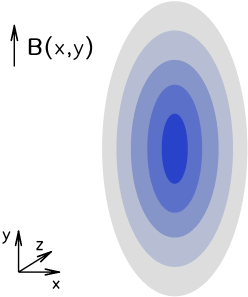


- red blood cells displaced more if they contain more (paramagnetic) haemoglobin [Okazaki et al '87]

Paramagnetism - heavy ions

- take a non-central heavy-ion collision
  $\hat{z}$: beam direction, $\hat{x}$-$\hat{y}$: transverse plane, $\hat{x}$: impact parameter





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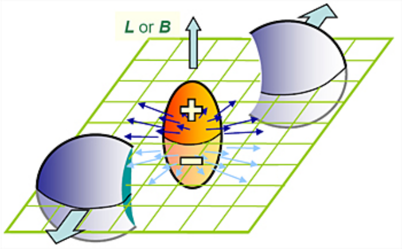

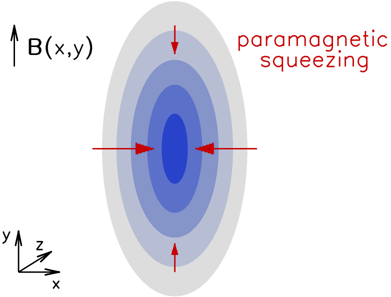

- free energy minimization squeezes QCD matter anisotropically
  [Bali,Bruckmann,Endrődi,Schäfer '13]

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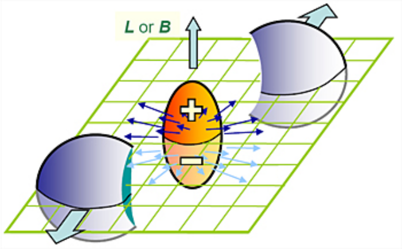
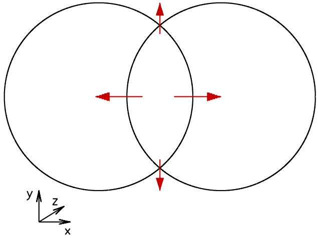
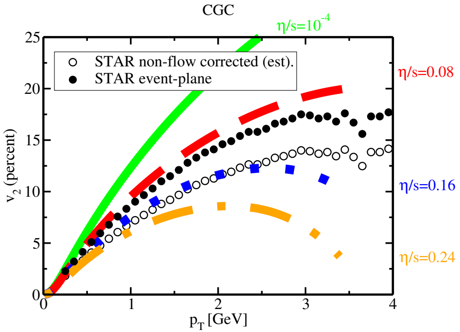




- elliptic flow: anisotropic pressure gradients due to initial geometry

Elliptic flow

- robust effect in non-central collisions
- integrated v_2 used to extract η/s



[Luzum, Romatschke '08]

Elliptic flow vs paramagnetic squeezing

- the force density produced through paramagnetism:

$$F^{\mathrm{ps}} = -\nabla \mathcal{F}^r = -\frac{\partial \mathcal{F}^r}{\partial (eB)} \cdot \nabla (eB) = M^r \cdot \nabla (eB).$$

- simplistic way to quantify it:

$$\Delta p'_{\mathrm{ps}} = F^{\mathrm{ps}}(\sigma_x, 0) - F^{\mathrm{ps}}(0, \sigma_y)$$

  - RHIC: $|\Delta p'_{\mathrm{ps}}| \approx 0.007\ \mathrm{GeV/fm}^4$
  - LHC: $|\Delta p'_{\mathrm{ps}}| \approx 0.7\ \mathrm{GeV/fm}^4$

- effect due to initial geometry
[Kolb et al '00; Petersen et al '06; Huovinen]

  - RHIC: $|\Delta p'_{\mathrm{g}}| \approx 0.1\ \mathrm{GeV/fm}^4$
  - LHC: $|\Delta p'_{\mathrm{g}}| \approx 1\ \mathrm{GeV/fm}^4$

Elliptic flow vs paramagnetic squeezing

- the force density produced through paramagnetism:

$$F^{\rm ps} = -\nabla \mathcal{F}^r = -\frac{\partial \mathcal{F}^r}{\partial (eB)} \cdot \nabla (eB) = M^r \cdot \nabla (eB).$$

- simplistic way to quantify it:

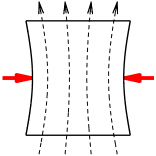
$$\Delta p'_{\rm ps} = F^{\rm ps}(\sigma_x, 0) - F^{\rm ps}(0, \sigma_y)$$

  - RHIC: $|\Delta p'_{\rm ps}| \approx 0.007\ \mathrm{GeV/fm}^4 \leftarrow \sim 7\%$ correction?
  - LHC: $|\Delta p'_{\rm ps}| \approx 0.7\ \mathrm{GeV/fm}^4 \leftarrow \sim 70\%$ correction?

- effect due to initial geometry
[Kolb et al '00; Petersen et al '06; Huovinen]
  - RHIC: $|\Delta p'_{\rm g}| \approx 0.1\ \mathrm{GeV/fm}^4$
  - LHC: $|\Delta p'_{\rm g}| \approx 1\ \mathrm{GeV/fm}^4$

Equation of state – conclusions

- at $B > 0$, pressure depends on the scheme (B vs Φ)



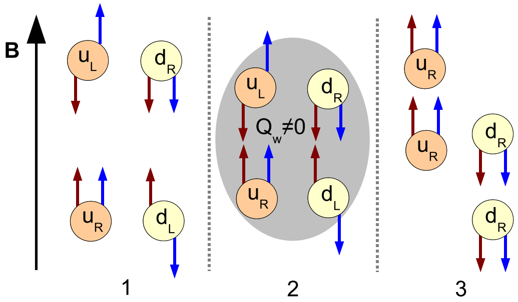
- QCD vacuum is paramagnetic (free quarks in SB limit; HRG; lattice QCD with physical m_π)
- at $T = 0$ this is a pure quantum effect; it produces no entropy
- paramagnetism + non-uniform fields = squeezing effect
  - in heavy-ion collisions: competes with elliptic flow
  - crude estimate: it might be important
  - future model descriptions: take into account $B(x, y, t)$ and compare the two effects more carefully

Outline - second part

- numerical results I: phase diagram
  - symmetries and order parameters
  - predictions from effective theories and models
  - magnetic catalysis and inverse catalysis
  - transition temperature, nature of transition at nonzero B
- numerical results II: equation of state
  - concept of the pressure in magnetic fields
  - magnetization, magnetic susceptibility
  - comparison to hadron resonance gas model
  - squeezing-effect in heavy-ion collisions
- numerical results III: chiral magnetic effect
  - electric polarization of CP-odd domains
  - comparison to model predictions

Chiral magnetic effect

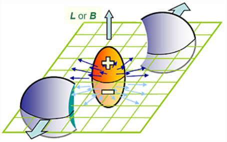
- local CP-violation through domains with $Q_{\mathrm{top}} \neq 0$?
- detect them through magnetic field B [Kharzeev et al. '08]

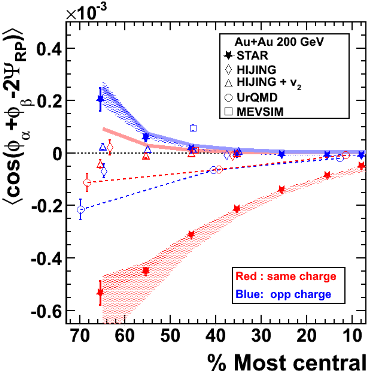


1. quarks interact with B: spins aligned
2. quarks interact with topology: chiralities (helicities) "aligned"
3. result: charge separation

Chiral magnetic effect

- $Q_{\rm top}$-domains fluctuate, direction of B fluctuates
  $\Rightarrow$ effect vanishes on average
- correlations may survive $(\alpha, \beta = \pm)$
  $a_{\alpha\beta} = -\cos\left[(\Phi_\alpha - \Psi_{\rm RP}) + (\Phi_\beta - \Psi_{\rm RP})\right]$





[STAR collaboration '09]

- need 3-particle correlations (technically complicated)
- CME prediction:
  $a_{++} = a_{--} = -a_{+-} > 0$
- CP-even backgrounds should be subtracted

Polarization of θ-domains

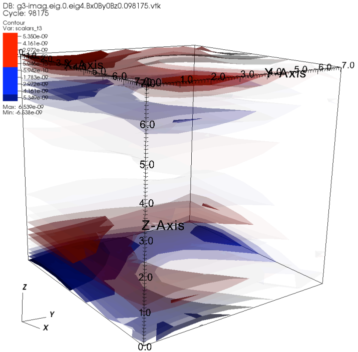
- magnetic/electric field induces magnetic/electric polarization

$$\bar{\psi}_f \sigma_{\mu\nu} \psi_f \propto q_f F_{\mu\nu}, \qquad \sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2i$$

- in the presence of topology, the roles are exchanged

$$\epsilon_{\mu\nu\alpha\beta}\, Q_{\rm top} \cdot \bar{\psi}_f \sigma_{\alpha\beta} \psi_f \propto q_f F_{\mu\nu}$$

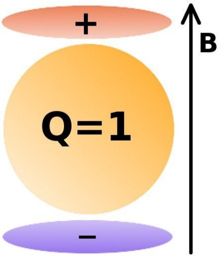
- put instanton configuration ($Q_{\rm top} = 1$) on the lattice and expose it to magnetic field ($B = F_{xy}$)
[Abramczyk et al '09]

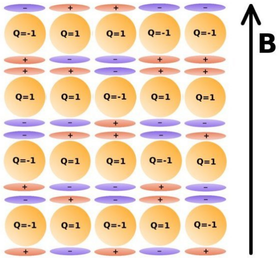


Polarization of θ-domains

- QCD vacuum: no instanton but local fluctuations in $q_{\rm top}$
- polarization exhibits a local correlation
  [Bali,Bruckmann,Endrődi,Fodor,Katz,Schäfer '14]

$$\left\langle \int \mathrm{d}^4 x\, q_{\rm top}(x) \cdot \bar{\psi}_f \sigma_{zt} \psi_f(x) \right\rangle \propto q_f F_{xy}$$





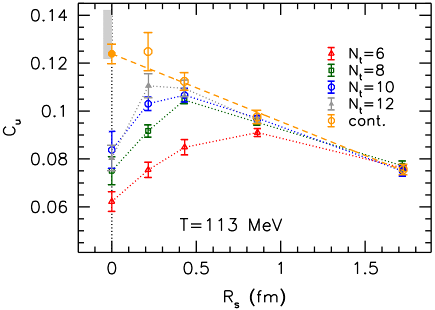
Polarization of θ-domains

- lattice approach
  - measure the correlator at physical m_π
  - renormalization involves smearing of fields over a range R_s
  - extrapolate to continuum limit and to $R_s \to 0$
- model description
  - assume $q_{\rm top}$ is generated by constant self-dual background fields $G_{xy} = G_{zt}$ (parallel to magnetic field F_{xy})
  - neglect quark masses $m^2 \ll F, G \leftrightarrow$ LLL approximation
  - assume normal distribution for $q_{\rm top}$
- consider the dimensionless combination

$$C_f = \frac{\left\langle q_{\rm top}(x) \cdot \bar{\psi}_f \sigma_{zt} \psi_f(x) \right\rangle}{\sqrt{\left\langle q_{\rm top}^2(x) \right\rangle} \left\langle \bar{\psi}_f \sigma_{xy} \psi_f(x) \right\rangle}$$

Polarization of θ-domains

- model: $C_f \sim 1 \Rightarrow B$-polarization equals E-polarization for unit topology
- lattice: $C_f \sim 0.13$

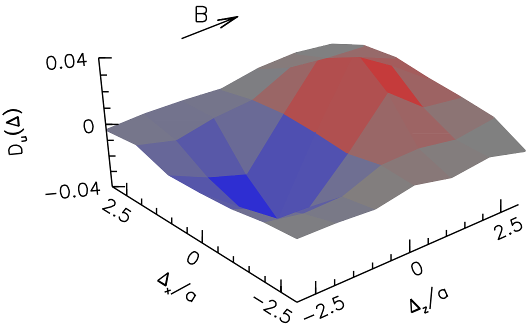


$\Rightarrow$ non-perturbative QCD interactions prevent full electric polarization of the quarks (for massive quarks spin flip becomes possible)

Electric charge separation

- the localized electric dipole moment is related to an extended charge structure

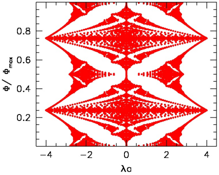
$$D_f(\Delta) = \left\langle \int \mathrm{d}^4x\, q_{\rm top}(x) \cdot \bar\psi_f \gamma_0 \psi_f(x+\Delta) \right\rangle \propto q_f B, \qquad \text{if } \Delta \parallel B$$

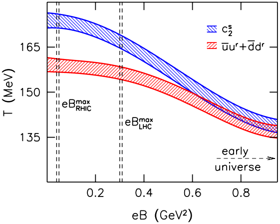


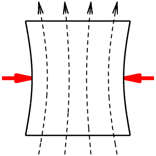
CME - conclusions

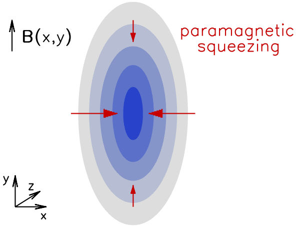
- local CP violation induced through B + fluctuating $q_{\mathrm{top}}(x)$
- usual assumptions (LLL, massless quarks) overestimate strength of local CP violation by order of magnitude

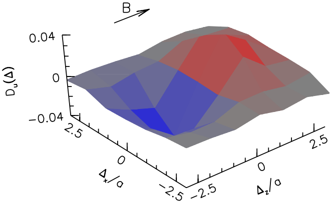
Summary


0.8
0.6
0.4
0.2
Φ / Φmax
−4
−2
0
2
4
λa


165
150
135
T (MeV)
0.2
0.4
0.6
0.8
eB (GeV²)
eBmax RHIC
eBmax LHC
early universe




B(x,y)
paramagnetic squeezing
y
z
x


B
0.04
0
−0.04
2.5
0
−2.5
−2.5
0
2.5